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Fuzziness in Partial Approximation Framework

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***Abstract*—In partial approximation spaces with Pawlakian approximation pairs, three partial membership functions are generated. These fuzzy functions rely on the lower and upper approximations of a set. They provide special type of fuzziness on the universe: all of them are partial functions and derived from the observed data relatively to available knowledge about the objects of the universe. With the help of these functions, three new approximation pairs are generated and so new approximation spaces appear effectively. Using not Pawlakian approximation pairs gives a special insight into the nature of general set approximations, and so new models of necessity and possibility can be given.**

I. Introduction

SET approximations were invented by Pawlak in the early 1980's which is known as rough set theory [1], [2], [3]. Its general scheme may be outlined as follows. Let a beforehand predefined family of subsets of the universe of objects be given. It is called the base system from which definable sets may be derived. Next, so-called lower and upper approximations can be formed with the help of definable sets via beforehand fixed rules in order to approximate any sets in the universe.

The starting point of rough set theory is a nonempty *finite* set U of objects and an equivalence relation ε on U [3]. The equivalence classes are called ε-elementary sets.

Definable sets are any unions of ε-elementary sets. Any set $S \subseteq U$ can be naturally approximated by the lower and upper ε-approximations of S which are denoted by $\underline{\varepsilon}$ and $\overline{\varepsilon}$, respectively. The former is the union of all ε-elementary sets which are the subsets of S, whereas the latter is the union of all ε-elementary sets which have a nonempty intersection with S.

A number of studies deal with the relationship between rough set theory and fuzzy set theory [4], [5], [6], [7], [8]. A detailed discussion of their connections and differences can be found, e.g., in [9], [10], [11].

There are many possibilities to establish a relationship between them [12], [13], [14], [15].

Just until now it has been generally accepted that the two theories are related but distinct and complementary to each other. Recently, however, Chakraborty has proposed a common ground relying on the *classical rough membership function* [16].

The classical rough membership function quantifies the degree of the relative overlap between a set $S \subseteq U$ and an ε-elementary set [10].[1] As usual, it is defined by

$$\mu_S^{\varepsilon}(u) = \frac{|[u]_\varepsilon \cap S|}{|[u]_\varepsilon|},$$

where $|\cdot|$ is the cardinality of a set, and $[u]_\varepsilon$ denotes the ε-elementary set to which a $u \in U$ belongs.

Hence, we just obtain a fuzzy membership function $\mu_S^{\varepsilon} : U \to [0,1]$ with

$$\begin{aligned}
\mu_S^{\varepsilon}(u) &= 1 \text{ if and only if } [u]_\varepsilon \subseteq S;\\
\mu_S^{\varepsilon}(u) &> 0 \text{ if and only if } [u]_\varepsilon \cap S \neq \emptyset;\\
\mu_S^{\varepsilon}(u) &= 0 \text{ if and only if } [u]_\varepsilon \cap S = \emptyset.
\end{aligned}$$

Thus, the rough membership function can be seen as a fuzzyfication of rough approximation, and μ_S^{ε} is a fuzzy subset of U induced by S.

One of the main features of μ_S^{ε} is that it relies on the system of base sets, the system of equivalence classes. In other words, *rough membership functions are generated by our knowledge* (appearing, e.g., in an information system). This is a distinctive feature of rough membership functions in contrast with fuzzy membership functions [18]. Furthermore, following from the definition of μ_S^{ε}, there are many constraints on the values of rough membership functions [12], [20], [21].

An important observation is that the Pawlakian lower and upper approximation pair can be reconstructed by employing the rough membership function. The well-known formulae are the following:

$$\begin{aligned}
\underline{\varepsilon}(S) &= \{u \in U \mid \mu_S^{\varepsilon}(u) = 1\},\\
\overline{\varepsilon}(S) &= \{u \in U \mid \mu_S^{\varepsilon}(u) > 0\}.
\end{aligned}$$

In the terminology of fuzzy set theory, the lower and upper approximations $\underline{\varepsilon}$ and $\overline{\varepsilon}$ are the *core* and the *support* of the fuzzy set μ_S^{ε}, respectively.

Nevertheless, Pawlakian set approximation has some very strong theoretical requirements:

- the system of base sets are total, i.e., their union gives back the universe;
- base sets are pairwise disjoint.

[1]Note that the notion of a classical rough membership function was explicitly introduced by Pawlak and Skowron in [10]. Nevertheless, it had been used and studied earlier by many authors. For more historical remarks, see [17]. Moreover, such a coefficient has already been considered by Łukasiewicz in 1913 [18], [19].

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In many cases, however, our knowledge does not fulfill these requirements:

- The partition shows the limit of our knowledge about the objects of the universe in the sense that two objects are indistinguishable if they belong to the same base set. On the other hand, it makes explicit our knowledge because we do distinguish two objects belonging to different base sets. Giving up the requirement of the pairwise disjoint property, the so–called *covering–based rough set theory* is obtained [22], [23], [24], [25], [26], [27].
- The universe may involve some objects without any information, i.e., base sets are not total. For instance, information systems often contain $NULL$ values. In the papers [28], [29], the authors give a very general system of the set approximation giving up both the pairwise disjoint property and the covering of the universe. It is called the (general) *partial approximation framework*.

In this paper, the above procedure is transferred to a partial set approximation context:

1) First, in a partial approximation space with a Pawlakian approximation pair, three *partial* membership functions are defined in the style of the classical rough membership function.
2) Then, three approximation pairs are generated with the help of partial membership functions. The question is whether these approximation pairs meet (at least) the minimum requirements of approximation pairs, i.e., these pairs actually form approximation pairs in partial approximation spaces.

The rest of the paper consists of three parts. In Section 2, the basic notions and notations of partial approximation spaces are summarized. In Section 3, three approximation pairs are generated as outlined above, and it is shown that they meet the minimum requirements prescribed for approximation pairs in partial approximation spaces. Section 4 consist of some remarks on the logical application of partial membership functions.

II. Partial approximation of sets

A. Basic notions and notations

Let U be a nonempty finite set and $\mathfrak{B} \subseteq 2^U$ be a nonempty family of nonempty subsets of U. U is the *universe of objects*, $\mathfrak{B}$ is the *base system* and its members are $\mathfrak{B}$*-sets* or *base sets* [30], [29], [31], [32], [33].

If $B \in \mathfrak{B}$ is a union of a family of sets $\mathfrak{B}' \subseteq \mathfrak{B} \setminus \{B\}$, B is called *reducible* in $\mathfrak{B}$, otherwise B is *irreducible* in $\mathfrak{B}$.

A base system $\mathfrak{B}$ is *single–layered* if every base set is irreducible, and *one–layered* if the base sets are pairwise disjoint. Of course, a one–layered base system is single–layered. From any base systems, single–layered and one–layered base systems can be constructed [31].

By formulae, a base system $\mathfrak{B}$ is single–layered, if

$$\forall B \in \mathfrak{B} \ \forall \mathfrak{B}' \subseteq \mathfrak{B} \setminus \{B\} \, (B \cap \bigcup \mathfrak{B}' \neq B),$$

and one–layered, if

$$\forall B \in \mathfrak{B} \ \forall \mathfrak{B}' \subseteq \mathfrak{B} \setminus \{B\} \, (B \cap \bigcup \mathfrak{B}' = \emptyset).$$

Informally, a base system $\mathfrak{B}$ is single–layered if every nonempty union of base sets has at least one member which belongs to exactly one base set, whereas $\mathfrak{B}$ is one–layered if all members of every nonempty union of base sets belong to exactly one base set.

During the approximation process, a family of sets $\mathfrak{D}_{\mathfrak{B}} \subseteq 2^U$ are applied. In the most general case, it is supposed only just that

1) $\mathfrak{D}_{\mathfrak{B}}$ is an extension of $\mathfrak{B}$, i.e., $\mathfrak{B} \subseteq \mathfrak{D}_{\mathfrak{B}}$;
2) $\emptyset \in \mathfrak{D}_{\mathfrak{B}}$.

Let $\mathsf{l}, \mathsf{u} : 2^U \to 2^U$ be an ordered pair of mappings and denoted it by $\langle \mathsf{l}, \mathsf{u} \rangle$.

The intended meaning of l and u is to express the lower and upper approximations of any subsets of U. Hence, it is called an *approximation pair*. The next definition specifies its *minimum requirements*.

Definition 1. An approximation pair $\langle \mathsf{l}, \mathsf{u} \rangle$ is a *weak approximation pair* if

(C0) $\mathsf{l}(2^U), \mathsf{u}(2^U) \subseteq \mathfrak{D}_{\mathfrak{B}}$ (*definability* of l and u);
(C1) l and u are monotone, i.e. for all $S_1, S_2 \in 2^U$ if $S_1 \subseteq S_2$ then $\mathsf{l}(S_1) \subseteq \mathsf{l}(S_2)$ and $\mathsf{u}(S_1) \subseteq \mathsf{u}(S_2)$ (*monotonicity* of l and u);
(C2) $\mathsf{u}(\emptyset) = \emptyset$ (*normality* of u);
(C3) if $S \subseteq U$, then $\mathsf{l}(S) \subseteq \mathsf{u}(S)$ (*weak approximation property*).

Clearly, l and u are many-to-one and $\mathsf{u}(2^U) \neq \mathsf{l}(2^U) \subseteq \mathfrak{D}_{\mathfrak{B}}$ in general.

Informally, definable sets represent our available knowledge about the the objects of the universe. They can be though of as *tools*, in more detail, base sets as *primary tools* and definable sets as *derived tools*. An approximation pair prescribes the *utilization* of tools in approximation processes.

It is reasonable that base sets as primary tools are exactly approximated from "lower side". In classical rough set theory, however, definable sets are exactly approximated from "lower side" as well.

Definition 2. A weak approximation pair $\langle \mathsf{l}, \mathsf{u} \rangle$ is

(C4) *granular* if $B \in \mathfrak{B}$, then $\mathsf{l}(B) = B$ (l is *granular*),
(C5) *standard* if $D \in \mathfrak{D}_{\mathfrak{B}}$, then $\mathsf{l}(D) = D$ (l is *standard*).

Of course, if l is standard, the granularity of l also holds.

The following proposition summarizes some simple consequences of the minimum requirements *(C0)–(C3)* in Definition 1 and the conditions *(C4)–(C5)* in Definition 2.

Proposition 1. *Let $\langle \mathsf{l}, \mathsf{u} \rangle$ be a weak approximation pair on U.*

1) $\mathsf{l}(\emptyset) = \emptyset$ *(normality of l).*
2) l *is idempotent, i.e.,* $\mathsf{l}(\mathsf{l}(S)) = \mathsf{l}(S)$ *for all* $S \in 2^U$*, and* $\mathsf{l}(2^U) = \mathfrak{D}_{\mathfrak{B}}$ *if and only if* l *is standard.*

3) a) *If* $\mathsf{l}(S) = S$*, then* $S \in \mathfrak{D}_{\mathfrak{B}}$.
   b) *Let* l *be standard. Then,* $\mathsf{l}(S) = S$ *if and only if* $S \in \mathfrak{D}_{\mathfrak{B}}$.
4) a) *If* $\mathsf{l}(U) = \bigcup \mathfrak{D}_{\mathfrak{B}}$*, then* $\bigcup \mathfrak{D}_{\mathfrak{B}} \in \mathfrak{D}_{\mathfrak{B}}$.
   b) *Let* l *be standard. Then,* $\mathsf{l}(U) = \bigcup \mathfrak{D}_{\mathfrak{B}}$ *if and only if* $\bigcup \mathfrak{D}_{\mathfrak{B}} \in \mathfrak{D}_{\mathfrak{B}}$.

The next definition deals with the question how lower and upper approximations relate to the approximated sets.

Definition 3. A weak approximation pair $\langle \mathsf{l}, \mathsf{u} \rangle$ is

(C6) *lower semi–strong* if $\mathsf{l}(S) \subseteq S$ for all $S \in 2^U$ (i.e., l is contractive);
(C7) *upper semi–strong* if $S \subseteq \mathsf{u}(S)$ for all $S \in 2^U$ (i.e., u is extensive);
(C8) *strong* if it is lower and upper semi–strong, i.e., each subset $S \in 2^U$ is bounded by $\mathsf{l}(S)$ and $\mathsf{u}(S)$: $\mathsf{l}(S) \subseteq S \subseteq \mathsf{u}(S)$.

Proposition 2.

1) *If* $\langle \mathsf{l}, \mathsf{u} \rangle$ *is an upper semi–strong approximation pair on* U*, then* $\mathsf{u}(U) = U$ *(co–normality of* u*).*
2) *If* $\langle \mathsf{l}, \mathsf{u} \rangle$ *is an upper semi–strong approximation pair on* U *and* l *is standard, then* $\mathsf{l}(U) = U$ *(co–normality of* l*).*

Based on the foregoing, a general set–theoretic partial approximation framework can be defined as follows.

Definition 4. The ordered 5–tuple $\mathsf{GAS}(U) = \langle U, \mathfrak{B}, \mathfrak{D}_{\mathfrak{B}}, \mathsf{l}, \mathsf{u} \rangle$ whose components are defined as before, is called a *(general) approximation space*.

Definition 5. $\mathsf{GAS}(U)$ is a *(general) total approximation space* or simply *total*, if $\mathfrak{B}$ covers the universe, i.e., $\bigcup \mathfrak{B} = U$, otherwise $\mathsf{GAS}(U)$ is a *(general) partial approximation space* or simply *partial*.

Definition 6. $\mathsf{GAS}(U)$ *relies on Pawlakian base*, if $\mathfrak{B}$ is a partition of U.

Corollary 1. $\mathsf{GAS}(U)$ *relies on Pawlakian base if and only if its base system is total and one–layered.*

Definition 7. The general approximation space $\mathsf{GAS}(U)$ is a *weak/standard/lower semi–strong/upper semi–strong/strong approximation space*, if the approximation pair $\langle \mathsf{l}, \mathsf{u} \rangle$ is weak/ standard/lower semi–strong/upper semi–strong/strong, respectively.

B. Exactness in general approximation spaces

In classical rough set theory, the notions of "crisp" and "definable" are inherently one and the same. In general approximation spaces, however, they can be differentiated.

Definition 8. Let $\mathsf{GAS}(U)$ be a weak approximation space and $S \subseteq U$.
S is *crisp*, if $\mathsf{l}(S) = \mathsf{u}(S)$, otherwise
S is *rough*.

If a set is crisp, its lower and upper approximations coincide with the approximated set only in strong approximation spaces. Furthermore, a crisp set is necessarily definable only in strong approximation spaces as well. However, it can easily be shown that a definable set is not necessarily crisp even in strong approximation spaces ([33], Example 8). Consequently, in general approximations spaces, the notions of "crisp" and "definability" are generally not synonymous to each other.

C. General approximation spaces with Pawlakian approximation pairs

Definition 9. $\mathsf{GAS}(U) = \langle U, \mathfrak{B}, \mathfrak{D}_{\mathfrak{B}}, \mathsf{l}, \mathsf{u} \rangle$ is a *approximation space with a Pawlakian approximation pair*, if

1) U is a finite nonempty set;
2) $\mathfrak{D}_{\mathfrak{B}}$ is *strict finite union type*, i.e., it is given by the following inductive definition:
   a) $\emptyset \in \mathfrak{D}_{\mathfrak{B}}$;
   b) $\mathfrak{B} \subseteq \mathfrak{D}_{\mathfrak{B}}$;
   c) if $B_1, B_2 \in \mathfrak{B}$, then $B_1 \cup B_2 \in \mathfrak{D}_{\mathfrak{B}}$;
3) $\langle \mathsf{l}, \mathsf{u} \rangle$ is a *Pawlakian approximation pair*, i.e.,
   a) $\mathsf{l}(S) = \bigcup \mathbb{L}(S)$, where $\mathbb{L}(S) = \{B \in \mathfrak{B} \mid B \subseteq S\}$;
   b) $\mathsf{u}(S) = \bigcup \mathbb{U}(S)$, where $\mathbb{U}(S) = \{B \in \mathfrak{B} \mid B \cap S \neq \emptyset\}$.

Proposition 3. *Let* $\mathsf{GAS}(U)$ *be an approximation space with a Pawlakian approximation pair.*

1) $\mathsf{GAS}(U)$ *is a standard lower semi–strong approximation space.*
2) $\mathsf{GAS}(U)$ *is an upper semi–strong approximation space if and only if* $\mathfrak{B}$ *covers the universe.*

Definition 10. Let $\mathsf{GAS}(U)$ be an approximation space with a Pawlakian approximation pair and $S \subseteq U$. Then

$$\mathsf{b}(S) = \bigcup (\mathbb{U}(S) \setminus \mathbb{L}(S))$$

is called the *boundary* of S.

Clearly, $\mathsf{b}(S) \subseteq \mathsf{u}(S)$ for all $S \subseteq U$.

Corollary 2. *Let* $\mathsf{GAS}(U)$ *be an approximation space with a Pawlakian approximation pair.*

1) *In general,* $\mathsf{u}(S) \setminus \mathsf{l}(S) \subseteq \mathsf{b}(S)$ *for any* $S \subseteq U$.
2) *If* $S \subseteq U$,

$$\mathsf{b}(S) = \mathsf{u}(S) \setminus \mathsf{l}(S) \Leftrightarrow \mathsf{b}(S) \cap \mathsf{l}(S) = \emptyset.$$

Proof:

1) $u \in \mathsf{u}(S) \setminus \mathsf{l}(S)$

$$\begin{aligned}
&\Leftrightarrow \quad u \in \bigcup \mathbb{U}(S) \wedge u \notin \bigcup \mathbb{L}(S) \\
&\Leftrightarrow \quad \exists B \in \mathfrak{B}\, (u \in B \wedge B \in \mathbb{U}(S) \wedge B \notin \mathbb{L}(S)) \\
&\Leftrightarrow \quad \exists B \in \mathfrak{B}\, (u \in B \wedge B \in \mathbb{U}(S) \setminus \mathbb{L}(S)) \\
&\Rightarrow \quad u \in \bigcup (\mathbb{U}(S) \setminus \mathbb{L}(S)) = \mathsf{b}(S)
\end{aligned}$$

2) $(\Rightarrow)$ $\mathsf{b}(S) \cap \mathsf{l}(S) = (\mathsf{u}(S) \setminus \mathsf{l}(S)) \cap \mathsf{l}(S) = \emptyset$.
   $(\Leftarrow)$ $\mathsf{b}(S)$

$$\begin{aligned} &= (\mathsf{b}(S) \cap \mathsf{l}(S)) \cup (\mathsf{b}(S) \cap (\mathsf{l}(S))^c) \\ &= \mathsf{b}(S) \cap (\mathsf{l}(S))^c \\ &\subseteq \mathsf{u}(S) \cap (\mathsf{l}(S))^c = \mathsf{u}(S) \setminus \mathsf{l}(S), \end{aligned}$$

which are compared to (1), we get

$$\mathsf{b}(S) = \mathsf{u}(S) \setminus \mathsf{l}(S).$$

■

III. Fuzziness in partial approximation spaces with Pawlakian approximation pairs

Let $\mathsf{GAS}(U) = \langle U, \mathfrak{B}, \mathfrak{D}_{\mathfrak{B}}, \mathsf{l}, \mathsf{u}\rangle$ be a partial approximation space with a Pawlakian approximation pair. In other words, $\mathsf{GAS}(U)$ is an approximation space with a Pawlakian approximation pair and $\bigcup \mathfrak{B} \subseteq U$.

A. Partial membership functions

If $u \in U$, let $\mathcal{N}_{\mathfrak{B}}(u) = \{B \in \mathfrak{B} \mid u \in B\}$. The family of sets $\mathcal{N}_{\mathfrak{B}}(u)$ may be called the *(reflexive) neighborhood system* of u with respect to the base system $\mathfrak{B}$ [34], and its members are called the *neighborhoods* of u.

Three different partial membership functions are defined in $\mathsf{GAS}(U)$ as follows [32], [35], [36], [38], [20].

Definition 11. Let $\mathsf{GAS}(U) = \langle U, \mathfrak{B}, \mathfrak{D}_{\mathfrak{B}}, \mathsf{l}, \mathsf{u}\rangle$ be a partial approximation space with a Pawlakian approximation pair and $S \subseteq U$.

$\mu_S^o, \mu_S^a, \mu_S^p : U \to [0,1]$ are *optimistic/average/pessimistic partial membership functions*, respectively, if

$$\mu_S^o(u) = \begin{cases} \mathsf{max}\left\{\frac{|B\cap S|}{|B|} \mid B \in \mathcal{N}_{\mathfrak{B}}(u)\right\}, & \text{if } u \in \bigcup \mathfrak{B}; \\ \mathsf{undefined}, & \text{otherwise}; \end{cases}$$
$$\mu_S^a(u) = \begin{cases} \mathsf{avg}\left\{\frac{|B\cap S|}{|B|} \mid B \in \mathcal{N}_{\mathfrak{B}}(u)\right\}, & \text{if } u \in \bigcup \mathfrak{B}; \\ \mathsf{undefined}, & \text{otherwise}; \end{cases}$$
$$\mu_S^p(u) = \begin{cases} \mathsf{min}\left\{\frac{|B\cap S|}{|B|} \mid B \in \mathcal{N}_{\mathfrak{B}}(u)\right\}, & \text{if } u \in \bigcup \mathfrak{B}; \\ \mathsf{undefined}, & \text{otherwise}. \end{cases}$$

Remark 1. For the sake of brevity, we will use the symbol "*" in order to denote a member of $\{o, a, p\}$.

In Definition 11, each partial membership function μ_S^* forms a special type of fuzziness on U which is induced by the base system $\mathfrak{B}$, i.e., our available knowledge (primary tools) about the objects of the universe.

An important feature of each μ_S^* is that it is a *partial* function. Clearly, if $\bigcup \mathfrak{B} \subsetneqq U$, $\mu_S^*(u)$ is undefinable for all $u \in U \setminus \bigcup \mathfrak{B}$. In other words, $\mathtt{dom}\, \mu_S^* = \bigcup \mathfrak{B} \subsetneqq U$.[2]

The following statements can easily be checked.

Proposition 4. *Let* $\mathsf{GAS}(U) = \langle U, \mathfrak{B}, \mathfrak{D}_{\mathfrak{B}}, \mathsf{l}, \mathsf{u}\rangle$ *be a partial approximation space with a Pawlakian approximation pair. Then, for any* $S \subseteq U$ *and* $u \in U$

1) $\mu_S^o(u) = 1$ *if and only if*
$$\exists B \in \mathcal{N}_{\mathfrak{B}}(u)\, (B \subseteq S) \ (\textit{i.e., } \mathcal{N}_{\mathfrak{B}}(u) \cap \mathbb{L}(S) \neq \emptyset);$$
2) $\mu_S^a(u) = 1$, $\mu_S^p(u) = 1$ *if and only if*
$$\forall B \in \mathcal{N}_{\mathfrak{B}}(u)\, (B \subseteq S) \ (\textit{i.e., } \mathcal{N}_{\mathfrak{B}}(u) \subseteq \mathbb{L}(S));$$
3) $\mu_S^o(u) > 0$, $\mu_S^a(u) > 0$ *if and only if*
$$\exists B \in \mathcal{N}_{\mathfrak{B}}(u)\, (B \cap S \neq \emptyset) \ (\textit{i.e., } \mathcal{N}_{\mathfrak{B}}(u) \cap \mathbb{U}(S) \neq \emptyset);$$
4) $\mu_S^p(u) > 0$ *if and only if*
$$\forall B \in \mathcal{N}_{\mathfrak{B}}(u)\, (B \cap S \neq \emptyset) \ (\textit{i.e., } \mathcal{N}_{\mathfrak{B}}(u) \subseteq \mathbb{U}(S));$$
5) $\mu_S^o(u), \mu_S^a(u) = 0$ *if and only if*
$$\forall B \in \mathcal{N}_{\mathfrak{B}}(u)\, (B \cap S = \emptyset) \ (\textit{i.e., } \mathcal{N}_{\mathfrak{B}}(u) \cap \mathbb{U}(S) = \emptyset).$$
6) $\mu_S^p(u) = 0$ *if and only if*
$$\exists B \in \mathcal{N}_{\mathfrak{B}}(u)\, (B \cap S = \emptyset).$$

Proposition 4 implies the following statements.

Corollary 3. *Let* $\mathsf{GAS}(U)$ *be a partial approximation space with a Pawlakian approximation pair. Then, for the optimistic partial membership function* μ_S^o,

1) $\mu_S^o(u) = 1$ *if and only if* $u \in \mathsf{l}(S)$,
2) $\mu_S^o(u) > 0$ *if and only if* $u \in \mathsf{u}(S)$,
3) $0 < \mu_S^o(u) < 1$ *if and only if* $u \in \mathsf{u}(S) \setminus \mathsf{l}(S)$,
4) $\mu_S^o(u) = 0$ *if and only if* $u \in \bigcup \mathfrak{B} \setminus \mathsf{u}(S)$,

for any $S \subseteq U$ *and* $u \in U$.

Corollary 4. *Let* $\mathsf{GAS}(U)$ *be a partial approximation space with a Pawlakian approximation pair. Then, for the average partial membership function* μ_S^a,

1) *if* $\mu_S^a(u) = 1$, *then* $u \in \mathsf{l}(S)$,
2) $\mu_S^a(u) > 0$ *if and only if* $u \in \mathsf{u}(S)$,
3) *if* $u \in \mathsf{u}(S) \setminus \mathsf{l}(S)$, *then* $0 < \mu_S^a(u) < 1$,
4) $\mu_S^a(u) = 0$ *if and only if* $u \in \bigcup \mathfrak{B} \setminus \mathsf{u}(S)$,

for any $S \subseteq U$ *and* $u \in U$.

Corollary 5. *Let* $\mathsf{GAS}(U)$ *be a partial approximation space with a Pawlakian approximation pair. Then, for the pessimistic partial membership function* μ_S^p,

1) *if* $\mu_S^p(u) = 1$ *then* $u \in \mathsf{l}(S)$,
2) *if* $\mu_S^p(u) > 0$, *then* $u \in \mathsf{u}(S)$,
3) *if* $\mu_S^p(u) > 0$ *and* $u \notin \mathsf{l}(S)$,

   then $u \in \mathsf{u}(S)$ *and* $\mu_S^p(u) < 1$,
4) *if* $u \in \bigcup \mathfrak{B} \setminus \mathsf{u}(S)$, *then* $\mu_S^p(u) = 0$.

for any $S \subseteq U$ *and* $u \in U$.

The different notions of *necessity* and *possibility* can be found in the definitions of partial membership functions μ_S^*.

The values $\mu_S^*(u)$ ($u \in U$) of the partial membership functions defined above informally mean the following.

The case of optimistic partial membership function:

1) if $\mu_S^o(u) = 1$, i.e., u has at least one neighborhood inside S, u can certainly be classified as belonging to S in an optimistic sense;

[2] $\mathtt{dom}\, f$ denotes the domain of the map f.

2) if $\mu_S^o(u) > 0$, i.e., u has at least one neighborhood wholly or partly inside S, u can possibly be classified as belonging to S in an optimistic sense;
3) if $0 < \mu_S^o(u) < 1$, i.e., u does not have any neighborhood inside S but has at least one neighborhood partly inside and partly outside S, u cannot be classified as either belonging to S or does not belonging to S in an optimistic sense.

The case of the average partial membership function:

1) if $\mu_S^a(u) = 1$, i.e., all neighborhoods of u are inside S, u can certainly be classified as belonging to S in average approach;
2) if $\mu_S^a(u) > 0$, i.e., u has at least one neighborhood wholly or partly inside S, u can possibly be classified as belonging to S in average approach;
3) if $0 < \mu_S^a(u) < 1$, i.e., u has a neighborhood not inside S and has at least one neighborhood wholly or partly inside S, u cannot be classified as either belonging to S or does not belonging to S in average approach.

The case of pessimistic partial membership function:

1) if $\mu_S^p(u) = 1$, i.e., all neighborhoods of u are inside S, u can certainly be classified as belonging to S in a pessimistic sense;
2) if $\mu_S^p(u) > 0$, i.e., all neighborhoods of u are wholly or partly inside S, u can possibly classified as belonging to S in a pessimistic sense;
3) if $0 < \mu_S^p(u) < 1$, i.e., u has a neighborhood not inside S and all neighborhoods of u are wholly or partly inside S, u cannot be classified as either belonging to S or does not belonging to S in a pessimistic sense.

Last, for all three partial membership functions,

$$\mu_S^*(u) = \mathsf{undefined}$$

indicates that we do not have any information about u. Consequently, defining membership degree for u should be meaningless with respect to our knowledge about the objects of the universe.

In classical rough set theory, lower and upper approximations and the boundary can be reconstructed setting out from the membership function. In a fuzzy context, the reconstruction can be carried out by means of `core` and `support` of membership functions in a standard way.

As usual, for the partial membership function μ_S^*, the `core` and `support` are the following:

$$\mathtt{core}(\mu_S^*) = \{u \in U \mid \mu_S^*(u) = 1\};$$
$$\mathtt{support}(\mu_S^*) = \{u \in U \mid \mu_S^*(u) > 0\}.$$

Now, $\mathsf{l}^*, \mathsf{u}^* : 2^U \to 2^U$ approximation pair may be defined as usual:

$$\begin{aligned} \mathsf{l}^*(S) &= \mathtt{core}(\mu_S^*) = \{u \in U \mid \mu_S^*(u) = 1\}, \\ \mathsf{u}^*(S) &= \mathtt{support}(\mu_S^*) = \{u \in U \mid \mu_S^*(u) > 0\}. \end{aligned}$$

1) The case of optimistic partial membership functions: In the case of the optimistic partial membership function μ_S^o, the *optimistic lower and upper approximation pair* is the following:

$$\begin{aligned} \mathsf{l}^o(S) &= \mathtt{core}(\mu_S^o) = \{u \in U \mid \mu_S^o(u) = 1\} \\ &= \{u \in \mathsf{l}(S) \mid \exists B \in \mathcal{N}_{\mathfrak{B}}(u)\,(B \subseteq S)\} \\ &= \mathsf{l}(S) \end{aligned}$$

by Corollary 3 (1), and

$$\begin{aligned} \mathsf{u}^o(S) &= \mathtt{support}(\mu_S^o) = \{u \in U \mid \mu_S^o(u) > 0\} \\ &= \{u \in \mathsf{u}(S) \mid \exists B \in \mathcal{N}_{\mathfrak{B}}(u)\,(B \cap S \neq \emptyset)\} \\ &= \mathsf{u}(S) \end{aligned}$$

by Corollary 3 (2).

Informally, $\mathsf{l}^o(S)$ is a collection of such $u \in U$ which has at least one neighborhood included in S, and $\mathsf{l}^o(S) = \mathsf{l}(S)$. $\mathsf{u}^o(S)$ is a collection of such $u \in U$ which has at least one neighborhood having nonempty intersection with S, and $\mathsf{u}^o(S) = \mathsf{u}(S)$.

In other words, in the case of optimistic partial membership function μ_S^o, we get back the Pawlakian approximation pair $\langle \mathsf{l}, \mathsf{u} \rangle$. It implies that $\langle \mathsf{l}^o, \mathsf{u}^o \rangle$ meets the minimum requirements *(C0)–(C3)* and the conditions *(C4)–(C5)*.

2) The case of average partial membership functions: In the case of the average partial membership function μ_S^o, the *average lower and upper approximation pair* is the following:

$$\begin{aligned} \mathsf{l}^a(S) &= \mathtt{core}(\mu_S^a) = \{u \in U \mid \mu_S^a(u) = 1\} \\ &= \{u \in U \mid \forall B \in \mathcal{N}_{\mathfrak{B}}(u)\,(B \subseteq S)\} \\ &\subseteq \mathsf{l}(S) \end{aligned}$$

by Corollary 4 (1), and

$$\begin{aligned} \mathsf{u}^a(S) &= \mathtt{support}(\mu_S^a) = \{u \in U \mid \mu_S^a(u) > 0\} \\ &= \{u \in \mathsf{u}(S) \mid \exists B \in \mathcal{N}_{\mathfrak{B}}(u)\,(B \cap S \neq \emptyset)\} \\ &= \mathsf{u}(S) \end{aligned}$$

by Corollary 4 (2).

Informally, $\mathsf{l}^a(S)$ is a collection of such a $u \in U$ whose all neighborhoods included in S, and $\mathsf{l}^a(S) \subseteq \mathsf{l}(S)$. $\mathsf{u}^a(S)$ is a collection of such a $u \in U$ which has at least one neighborhood having nonempty intersection with S, and $\mathsf{u}^a(S) = \mathsf{u}(S)$.

That is, in the case of average partial membership function μ_S^a, we get back the upper Pawlakian approximation map, but the Pawlakian lower approximation map has already changed.

Proposition 5. $\mathsf{GAS}(U) = \langle U, \mathfrak{B}, \mathfrak{D}_{\mathfrak{B}}^a, \mathsf{l}^a, \mathsf{u}^a \rangle$ *is a weak general approximation space provided that* $\mathfrak{D}_1 \setminus \mathfrak{D}_2 \in \mathfrak{D}_{\mathfrak{B}}^a$ *(*$\mathfrak{D}_1, \mathfrak{D}_2 \in \mathfrak{D}_{\mathfrak{B}}$*).*

Proof:

(C0)–(C2) They are straightforward.

(C3) If $u \in \mathsf{l}^a(S)$, then $\forall B \in \mathcal{N}_{\mathfrak{B}}(u)\,(B \subseteq S)$, and so $\exists B \in \mathcal{N}_{\mathfrak{B}}(u)\,(B \cap S \neq \emptyset)$, i.e., $u \in \mathsf{u}^a(S)$.

■

3) The case of pessimistic partial membership functions: In the case of the pessimistic partial membership function μ_S^p, the *pessimistic lower and upper approximation pair* is the following:

$$\begin{aligned} \mathsf{l}^p(S) &= \mathtt{core}(\mu_S^p) = \{u \in U \mid \mu_S^p(u) = 1\} \\ &= \{u \in U \mid \forall B \in \mathcal{N}_{\mathfrak{B}}(u)\,(B \subseteq S)\} \\ &\subseteq \mathsf{l}(S) \end{aligned}$$

by Corollary 5 (1), and

$$\begin{aligned} \mathsf{u}^p(S) &= \mathtt{support}(\mu_S^p) = \{u \in U \mid \mu_S^p(u) > 0\} \\ &= \{u \in U \mid \forall B \in \mathcal{N}_{\mathfrak{B}}(u)\,(B \cap S \neq \emptyset)\} \\ &\subseteq \mathsf{u}(S) \end{aligned}$$

by Corollary 5 (2).

Informally, $\mathsf{l}^p(S)$ is a collection of such $u \in U$ whose all neighborhoods included in S, and $\mathsf{l}^p(S) \subseteq \mathsf{l}(S)$. $\mathsf{u}^p(S)$ is a collection of such $u \in U$ whose all neighborhoods having nonempty intersection with S, and $\mathsf{u}^p(S) \subseteq \mathsf{u}(S)$.

In the case of pessimistic partial membership function μ_S^p, both lower and upper Pawlakian approximation maps have changed.

Proposition 6. *$\mathsf{GAS}(U) = \langle U, \mathfrak{B}, \mathfrak{D}_{\mathfrak{B}}^p, \mathsf{l}^p, \mathsf{u}^p\rangle$ is a weak general approximation space provided that $\mathfrak{D}_1 \setminus \mathfrak{D}_2 \in \mathfrak{D}_{\mathfrak{B}}^p$ ($\mathfrak{D}_1, \mathfrak{D}_2 \in \mathfrak{D}_{\mathfrak{B}}$).*

Proof:

(C0)–(C2) They are straightforward.

(C3) If $u \in \mathsf{l}^p(S)$, then $\forall B \in \mathcal{N}_{\mathfrak{B}}(u)\,(B \subseteq S)$, and so $\forall B \in \mathcal{N}_{\mathfrak{B}}(u)\,(B \cap S \neq \emptyset)$, i.e., $u \in \mathsf{u}^p(S)$. ■

The next proposition deals with the conditions *(C4)–(C5)* of average and pessimistic approximation pairs.

Proposition 7. *Let $\langle U, \mathfrak{B}, \mathfrak{D}_{\mathfrak{B}}^a, \mathsf{l}^a, \mathsf{u}^a\rangle$ and $\langle U, \mathfrak{B}, \mathfrak{D}_{\mathfrak{B}}^p, \mathsf{l}^p, \mathsf{u}^p\rangle$ be weak approximation spaces whose components are defined as above.*

If the base system $\mathfrak{B}$ is one–layered, $\mathfrak{D}_{\mathfrak{B}}^a = \mathfrak{D}_{\mathfrak{B}}^p = \mathfrak{D}_{\mathfrak{B}}$ and the weak approximation pairs $\langle \mathsf{l}^a, \mathsf{u}^a\rangle$ and $\langle \mathsf{l}^p, \mathsf{u}^p\rangle$ are standard, i.e., $\mathsf{l}^a(D) = D$ and $\mathsf{l}^p(D) = D$ for all $D \in \mathfrak{D}_{\mathfrak{B}}$.

Proof:

Since l is standard, $\mathsf{l}^a(D) \subseteq \mathsf{l}(D) = D$ for all $D \in \mathfrak{D}_{\mathfrak{B}}$.

On the other hand, $\mathfrak{B}$ is one–layered, and so every definable set $D \in \mathfrak{D}_{\mathfrak{B}}$ is a finite union of pairwise disjoint base sets, e.g., $D = B_1 \cup \cdots \cup B_n$, where B_i's are pairwise disjoint. Moreover, for every $u \in D$ there exists exactly one $i \in \{1, 2, \ldots, n\}$ in such a way that $\mathcal{N}_{\mathfrak{B}}(u) = \{B_i\}$.

Hence, we get for all $D \in \mathfrak{D}_{\mathfrak{B}}$,

$$\begin{aligned} \mathsf{l}^a(D) &= \{u \in U \mid \forall B \in \mathcal{N}_{\mathfrak{B}}(u)\,(B \subseteq D)\} \\ &\supseteqq \{u \in D \mid \forall B \in \mathcal{N}_{\mathfrak{B}}(u)\,(B \subseteq D)\} \\ &= \{u \in B_1 \cup \cdots \cup B_n \mid \forall B \in \mathcal{N}_{\mathfrak{B}}(u)\,(B \subseteq D)\} \\ &= \{u \in B_1 \mid \forall B \in \mathcal{N}_{\mathfrak{B}}(u)\,(B \subseteq D)\} \\ &\quad \cup \cdots \cup \{u \in B_n \mid \forall B \in \mathcal{N}_{\mathfrak{B}}(u)\,(B \subseteq D)\} \\ &= B_1 \cup \cdots \cup B_n = D. \end{aligned}$$

Therefore, $\mathsf{l}^a(D) = D$.

The standard property of l^p can be proved similarly. ■

IV. Some remarks on the logical applications

In the previous sections, first, three partial membership functions have been defined in partial approximation spaces with Pawlakian approximation pairs, then three approximation pairs have been generated with the help of them. It has been shown that, among others, they meet the minimum requirements prescribed for approximation pairs in partial approximation spaces, i.e., they actually form approximation pairs.

Optimistic, average and pessimistic partial membership functions have already been studied by the second author from the logical point of view in [38], [32]. It turned out that they are in connection with *decision–theoretic rough set models* (DTRS) which can be considered as the probabilistic extensions of algebraic rough set models [37].

Optimistic, average and pessimistic partial membership functions may serve as a bases of the semantics of a partial first–order logic. In the paper [35], the semantic system of a partial first–order logic with three different types of partial membership functions is presented. The proposed logical system gives an exact possibility to introduce different semantic notions of logical consequence relations which can be used in order to make clear the consequences of our decisions.

V. Conclusion and future work

In this paper, having defined three partial membership functions, three approximation pairs have been generated in partial approximation spaces with Pawlakian approximation pairs. We have investigated how these pairs meet the requirements prescribed for approximation pairs in partial approximation spaces. As a result, in this way we have constructed two not Pawlakian approximation pairs.

In the future, it should be worth performing similar investigations in partial approximation spaces setting out from *arbitrary* approximation pairs, in particular, which have been obtained in this paper.

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References

[1] Z. Pawlak, “Information systems theoretical foundations,” *Information Systems*, vol. 6, no. 3, pp. 205–218, 1981.
[2] ——, “Rough sets,” *International Journal of Computer and Information Sciences*, vol. 11, no. 5, pp. 341–356, 1982.
[3] ——, *Rough Sets: Theoretical Aspects of Reasoning about Data*. Kluwer Academic Publishers, Dordrecht, 1991.
[4] L. A. Zadeh, “Fuzzy sets,” *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[5] R. R. Yager, S. Ovchinnikov, R. M. Tong, and H. T. Nguyen, Eds., *Fuzzy sets and applications*. New York, NY, USA: Wiley-Interscience, 1987.

[6] B. Yuan and G. J. Klir, Eds., *Fuzzy Sets, Fuzzy Logic, and Fuzzy Systems: Selected Papers by Lotfi A. Zadeh*. River Edge, NJ, USA: World Scientific Publishing Co., Inc., 1996.

[7] G. J. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic, Theory and Applications*. New Jersey: Prentice Hall, 1995.

[8] D. Dubois and H. Prade, *Fuzzy sets and systems - Theory and applications*. New York: Academic press, 1980.

[9] Y. Y. Yao, "A comparative study of fuzzy sets and rough sets," *Information Sciences*, vol. 109, pp. 21–47, 1998.

[10] Z. Pawlak and A. Skowron, "Rough membership functions," in *Advances in the Dempster-Shafer theory of evidence*, R. R. Yager, J. Kacprzyk, and M. Fedrizzi, Eds. New York, NY, USA: John Wiley & Sons, Inc., 1994, pp. 251–271.

[11] L. Polkowski, *Rough Sets: Mathematical Foundations*, ser. Advances in Intelligent and Soft Computing. Physica-Verlag, A Springer-Verlag Company, 2002.

[12] J. Komorowski, Z. Pawlak, L. Polkowski, and A. Skowron, "Rough sets: A tutorial," in *Rough Fuzzy Hybridization. A New Trend in Decision-Making*, S. Pal and A. Skowron, Eds. Singapore: Springer-Verlag, 1999, pp. 3–98.

[13] D. Dubois and H. Prade, "Rough fuzzy sets and fuzzy rough sets," *Fuzzy Sets and Systems*, vol. 23, pp. 3–18, 1987.

[14] ——, "Rough fuzzy sets and fuzzy rough sets," *International Journal of General Systems*, vol. 17, no. 2-3, pp. 191–209, 1990.

[15] ——, "Putting rough sets and fuzzy sets together," in *Intelligent Decision Support - Handbook of Applications and Advances of the Rough Set Theory*, R. Slowinski, Ed. Kluwer Academic, Dordrecht, 1992, pp. 203–232.

[16] M. Chakraborty, "On fuzzy sets and rough sets from the perspective of indiscernibility," in *Logic and Its Applications. 4th Indian Conference, ICLA 2011 Delhi, India, January 5-11, 2011, Proceedings*, ser. LNAI, M. Banerjee and A. Seth, Eds., vol. 6521. Berlin Heidelberg: Springer-Verlag, 2011, pp. 22–37.

[17] Y. Yao, "Probabilistic rough set approximations," *International Journal of Approximate Reasoning*, vol. 49, no. 2, pp. 255–271, 2008.

[18] Z. Pawlak, L. Polkowski, and A. Skowron, "Rough sets: An approach to vagueness," in *Encyclopedia of Database Technologies and Applications*, L. C. Rivero, J. H. Doorn, and V. E. Ferraggine, Eds. Hershey, PA: Idea Group Inc., 2005, pp. 575–580.

[19] J. Łukasiewicz, "Die logischen grundlagen der wahrscheinlichkeitsrechnung (1913)," in *Jan Łukasiewicz - Selected Works*, L. Borkowski, Ed. Amsterdam, Warsaw: Polish Scientific Publishers and North-Holland Publishing Company, 1970.

[20] Y. Yao, "Semantics of fuzzy sets in rough set theory," in *Transactions on Rough Sets II*, ser. LNCS, J. F. Peters, A. Skowron, D. Dubois, J. W. Grzymała-Busse, M. Inuiguchi, and L. Polkowski, Eds. Springer Berlin Heidelberg, 2005, vol. 3135, pp. 297–318.

[21] A. Skowron and J. Stepaniuk, "Tolerance approximation spaces," *Fundamenta Informaticae*, vol. 27, no. 2-3, pp. 245–253, 1996.

[22] Z. Bonikowski, E. Bryniarski, and U. Wybraniec-Skardowska, "Extensions and intentions in the rough set theory," *Information Sciences*, vol. 107, no. 1-4, pp. 149–167, 1998.

[23] Y. Y. Yao, "On generalizing rough set theory," in *Proceedings of RSFDGrC 2003*, ser. LNAI 2639. Berlin Heidelberg: Springer-Verlag, 2003, pp. 44–51.

[24] Z. Pawlak and A. Skowron, "Rough sets: Some extensions," *Information Sciences*, vol. 177, pp. 28–40, 2007.

[25] W. Zhu and F.-Y. Wang, "On three types of covering-based rough sets," *IEEE Trans. Knowl. Data Eng.*, vol. 19, no. 8, pp. 1131–1144, 2007.

[26] W. Zhu, "Topological approaches to covering rough sets," *Information Sciences*, vol. 177, no. 6, pp. 1499–1508, 2007.

[27] ——, "Relationship between generalized rough sets based on binary relation and covering," *Information Sciences*, vol. 179, no. 3, pp. 210–225, 2009.

[28] Z. Csajbók and T. Mihálydeák, "Partial approximative set theory: A generalization of the rough set theory," *International Journal of Computer Information Systems and Industrial Management Applications*, vol. 4, pp. 437–444, 2012.

[29] ——, "A general set theoretic approximation framework," in *Proceedings of IPMU 2012, Catania, Italy, July 9-13, 2012, Part I*, ser. CCIS, S. Greco, B. Bouchon-Meunier, G. Coletti, M. Fedrizzi, B. Matarazzo, and R. R. Yager, Eds., vol. 297. Berlin Heidelberg: Springer-Verlag, 2012, pp. 604–612.

[30] Z. Csajbók, "Approximation of sets based on partial covering," *Theoretical Computer Science. Special Issue on Rough Sets and Fuzzy Sets in Natural Computing*, vol. 412, no. 42, pp. 5820–5833, 2011.

[31] Z. E. Csajbók, "Approximation of sets based on partial covering," in *Transactions on Rough Sets*, ser. LNCS, Transactions on Rough Sets XVI, J. F. Peters, A. Skowron, S. Ramanna, Z. Suraj, and X. Wang, Eds., vol. 7736. Heidelberg: Springer, 2013, pp. 144–220.

[32] T. Mihálydeák, "Partial first–order logic with approximative functors based on properties," in *Rough Sets and Knowledge Technology. 7th International Conference, RSKT 2012, Chengdu, China, August 17-20, 2012, Proceedings*, ser. LNAI, T. Li, H. S. Nguyen, G. Wang, J. Grzymala-Busse, R. Janicki, A. E. Hassanien, and H. Yu, Eds., vol. 7414. Berlin Heidelberg: Springer-Verlag, 2012, pp. 514–523.

[33] Z. Csajbók and T. Mihálydeák, "Partial approximative set theory: A generalization of the rough set theory," *International Journal of Computer Information Systems and Industrial Management Applications*, vol. 4, pp. 437–444, 2012.

[34] Y. Y. Yao, "Granular computing using neighborhood systems," in *Advances in Soft Computing: Engineering Design and Manufacturing. The 3rd On-line World Conference on Soft Computing (WSC3)*, R. Roy, T. Furuhashi, and P. K. Chawdhry, Eds. London: Springer-Verlag, 1999, pp. 539–553.

[35] T. Mihálydeák, "Partial first–order logic relying on optimistic, pessimistic and average partial membership functions." in *Proccedings of the 8th conference of the European Society for Fuzzy Logic and Technology, EUSFLAT-2013*, in the fall of 2013, Forthcoming.

[36] Y. Y. Yao and J. P. Zhang, "Interpreting fuzzy membership functions in the theory of rough sets," in *Rough Sets and Current Trends in Computing*, ser. Lecture Notes in Computer Science, W. Ziarko and Y. Y. Yao, Eds., vol. 2005. Springer, 2000, pp. 82–89.

[37] Y. Y. Yao, "Decision-theoretic rough set models," in *Rough Sets and Knowledge Technology. Second International Conference, RSKT 2007, Toronto, Canada, May 14-16, 2007. Proceedings*, ser. Lecture Notes in Computer Science, J. T. Yao et al., Eds., vol. 4481. Springer, 2007, pp. 1–12.

[38] T. Mihálydeák, "Partial firstorder logical semantics based on approximations of sets," in *Non-classical Modal and Perdicate Logics 2011, Guangzhou (Canton), China, F solutions, Prague*, P. Cintula, S. Ju, and M. Vita, Eds., 2011, pp. 85–90.