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An efficient algorithm for the density Turán problem of some unicyclic graphs

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***Abstract*—Let $H = (V(H), E(H))$ be a simple connected graph of order n with the vertex set $V(H)$ and the edge set $E(H)$. We consider a blow-up graph $G[H]$.**

We are interested in the following problem. We have to decide whether there exists a blow-up graph $G[H]$, with edge densities satisfying special conditions (homogeneous or inhomogeneous), such that the graph H does not appear in a blow-up graph as a transversal.

We study this problem for unicyclic graphs H with the cycle C_3. We show an efficient algorithm to decide whether a given set of edge densities ensures the existence of H in the blow-up graph $G[H]$.

***Index Terms*—blow-up graph; density; Turán density problem; unicyclic graph.**

I. Introduction

TURÁN [10] stated the first results in extremal graph theory. Then many authors extended this subject and formulated similar and new Turán density problems. [1], [3], [4], [6], [8], [9] and [11] obtained interesting results for some families of graphs.

In this paper we present an algorithm for testing whether a unicyclic graph with a given set of edge densities is a factor (transversal) of a blow-up graph. Our algorithm has the time complexity at most $\mathcal{O}(n^2)$, where n is the number of vertices of the unicyclic graph.

Csikvári and Nagy [5] discovered some interesting algorithm for testing whether a tree with a given set of edge densities is a factor of a blow-up graph. We extend their algorithm to the family of unicyclic graphs with the cycle C_3.

Now we define some notions and notations. Other definitions one can find in [2] and [7].

Let $H = (V(H), E(H))$ be a simple connected graph of order n with the vertex set $V(H)$ and the edge set $E(H)$. By P_k we denote the path with k vertices. By C_k we denote the cycle with k vertices. The set $S \subset V(H)$ is called *an independent vertex set* if the subgraph of H induced by S has empty set of edges.

Let

$$N_H(v) = \{x \in V(H) \mid \{v, x\} \in E(H)\}$$

be *the neighbourhood* of the vertex $v \in V(H)$ in the graph H. $|N_H(v)|$ is called *the degree* of v in $V(H)$. Each vertex of degree 1 in a graph H is called *a leaf* of the graph H.

We say that the graph H is *r-regular* if each vertex of H has degree r. The set $M \subset E(H)$ is called *the matching* (or *independent edge set*) in the graph H if the subgraph of H induced by M is 1-regular.

For a connected graph H we define *a blow-up graph* $G[H]$ of the graph H as follows. First we replace each vertex $i \in V(H)$ by an independent set of vertices A_i. Throughout this paper A_i is called *a cluster*. Next we connect vertices between the clusters A_i and A_j if i and j are adjacent in H, $i, j \in V(H)$. The graph induced by $A_i \cup A_j$ in $G[H]$ is a subgraph of a complete bipartite graph. See Fig. 2 and Fig. 3 which present examples of a blow-up graphs $G[H]$ of the graph H presented in Fig. 1.

For any two clusters we define *the density* between them by the following formula

$$d(A_i, A_j) = \frac{e(A_i, A_j)}{|A_i||A_j|},$$

where $e(A_i, A_j)$ denotes the number of edges between the clusters A_i and A_j.

The graph H is *a transversal* of $G[H]$ if H is a subgraph of $G[H]$ such that we have a homomorphism

$$\phi : V(H) \to V(G[H])$$

for which $\phi(i) \in A_i$ for all $i \in V(H)$. Other terminology: H is *a factor* of $G[H]$. An edge $e = \{i, j\}$ of the graph H we denote by $e = ij$.

The density Turán problem can be defined as follows. Let us determine the critical edge density, denoted by d_{crit}, which ensures the existence of the subgraph H of $G[H]$ as a transversal. Precisely, assume that all edges $e = \{i, j\}$ in the graph H satisfy the condition

$$d(A_i, Aj) > d_{crit},$$

where $i, j \in V(H)$. Then, no matter how we construct the blow-up graph $G[H]$, it contains the graph H as a transversal.

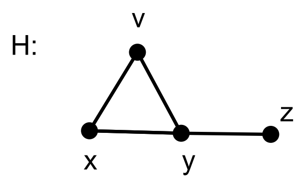

Fig. 1. The graph $H \in \mathcal{U}_{n,3}$ with the vertex set $V(H) = \{x, y, z, v\}$.

On the other words, for any value $d < d_{crit}$ there exists a blow-up graph $G[H]$ such that

$$d(A_i, A_j) > d$$

for all edges $ij \in E(H)$ and it does not contain H as a transversal. This problem was studied in [9].

By [5] we know that it is useful to consider more general problem. Let us assume that for every edge $e \in E(H)$ a density γ_e is given. Now our task is to decide if the set of densities $\{\gamma_e\}_{e \in E(H)}$ ensure the existence of the graph H as a transversal or we can construct a blow-up graph $G[H]$ such that

$$d(A_i, A_j) \geq \gamma_{ij},$$

but it does not induce the graph H as a transversal. This more general setting allows to use inductive proofs (see the proof of Theorem 7). We call this general case as *the inhomogeneous condition* on the edge densities, while the above condition of having a common lower bound $d_{crit}(H)$ for densities is called *the homogeneous case*.

Let $\mathcal{U}_{n,p}$ be a family of unicyclic graphs of order n with the cycle C_p. The path P_2 and the cycle C_3 are trivial unicyclic graphs for further considerations. In this paper we study the inhomogeneous density Turán problem for unicyclic graphs in the family $\mathcal{U}_{n,3}$, i.e. with the unique cycle C_3 (see Fig. 1).

Fig. 2 and Fig. 3 present two blow-up graphs $G_1[H]$ and $G_2[H]$ of the graph H presented in Fig. 1. In both cases we have the following values of the densities between clusters

$$d(A_x, A_y) = d(A_y, A_z) = \frac{3}{20},$$

$$d(A_x, A_v) = \frac{3}{16},$$

$$d(A_y, A_v) = \frac{1}{10}.$$

Let us recall the definition of *the multivariate matching polynomial* of the graph. The polynomial is the useful tool for the proof of our results.

Definition 1. *Let H be a graph and let $\underline{x_e}$ be the vector of variables x_e, $e \in E(H)$. We define the multivariate matching polynomial F_H of the graph H as follows*

$$F_H(\underline{x_e}, t) = \sum_{M \in \mathcal{M}} \left(\prod_{e \in M} x_e \right) (-t)^{|M|},$$

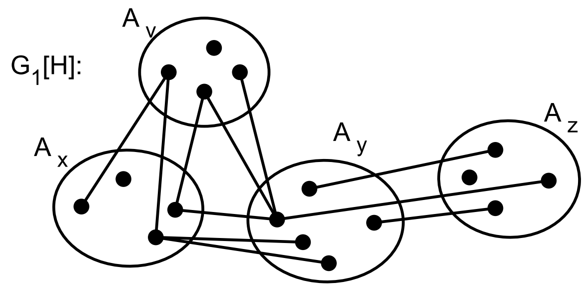

Fig. 2. An example of the blow-up graph $G[H]$ of the graph H presented in Fig. 1 with a transversal H.

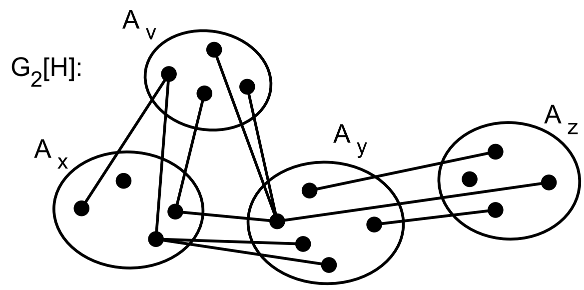

Fig. 3. An example of the blow-up graph $G[H]$ of the graph H presented in Fig. 1 without a transversal H.

where the summation goes over all matchings of the graph H, including the empty matching.

Fig. 4 and Fig. 5 present the paths P_2, P_4 and the unicycle graph $H \in \mathcal{U}_{6,3}$ with variables x_e assigned to each edge.

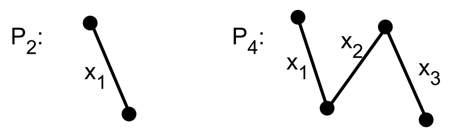

Fig. 4. Paths P_2 and P_4 with variables x_e assigned to each edge.

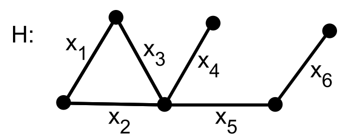

Fig. 5. Graph $H \in \mathcal{U}_{6,3}$ with variables x_e assigned to each edge.

By definition of the multivariate matching polynomial we have

$$F_{P_2}(\underline{x_e}, s) = 1 - sx_1,$$

$$F_{P_4}(\underline{x_e}, s) = 1 - s(x_1 + x_2 + x_3) + s^2 x_1 x_3,$$

$$F_H(\underline{x_e}, s) = 1 - s(x_1 + x_2 + x_3 + x_4 + x_5 + x_6) +$$
$$s^2(x_1x_4 + x_1x_5 + x_1x_6 + x_2x_6 + x_3x_6 + x_4x_6) - s^3 x_1x_4x_6.$$

II. Some results for the homogeneous case

For the completness of this paper we present some results for the homogeneous Turán density problem in this section. For this case Nagy [9] presented the following lower and upper bounds for the critical density d_{crit}.

Theorem 1 (Nagy [9]). *For a graph H we have*

$$\left(1-\frac{1}{\Delta(H)}\right) \leq d_{crit}(H) \leq \left(1-\frac{1}{\Delta^2(H)}\right),$$

where $\Delta(H)$ is the maximal degree of H.

Then Csikvári and Nagy [5] improved the upper bound.

Theorem 2 (Csikvári and Nagy [5]). *Let $\Delta(H)$ be the largest degree of the graph H. Then we have*

$$d_{crit}(H) \leq 1-\frac{1}{e(2\Delta(H)-1)},$$

where e is the base of the natural logarithm.

Now let us recall the definition of *the matching polynomial* of the graph.

Definition 2. *Let H be a weighted graph with constant weight function $w(e)=1$ for all edges $e \in E(H)$. Then the matching polynomial is defined as*

$$M(H,t)=\sum_{k=0}^{n/2}(-1)^k m_k(H)t^{n-2k},$$

where $m_k(H)$ denotes the number of k independent edges in the graph H.

Using this polynomial Csikvári and Nagy [5] stated the upper bound for the critical density as in Theorem 3.

Theorem 3 (Csikvári and Nagy [5]). *Let $\Delta(H)$ be the largest vertex degree in the graph H and let $t(H)$ be the largest root of the matching polynomial. Then we have*

$$d_{crit}(H) \leq 1-\tfrac{1}{(t(H))^2}.$$

In particular,

$$d_{crit}(H) < 1-\frac{1}{4(\Delta(H)-1)}.$$

What is more Nagy [9] showed the exact value of the critical density for trees.

Theorem 4 (Nagy [9]). *Let T be a tree. Then we have*

$$d_{crit}(T)=1-\frac{1}{\lambda^2_{max}(T)},$$

where $\lambda_{max}(T)$ is the maximum eigenvalue of the adjacency matrix of the tree.

Furthermore, Nagy [9] showed that for the cycle of order n and for the path of order $n+1$ the critical densities are equal.

Theorem 5 (Nagy [9]). *Let C_n be a cycle on n vertices and P_{n+1} be a path on $n+1$ vertices. Then we have*

$$d_{crit}(C_n)=d_{crit}(P_{n+1})=1-\frac{1}{4\cos^2\frac{\pi}{n+2}}.$$

We formulate the followig open problem.

Open problem: count the critical density $d_{crit}(H)$ for $H \in \mathcal{U}_{n,p}$, $p \geq 3$.

III. Inhomogeneous case: unicyclic graphs with the cycle C_3

In this section we study the inhomogeneous case when graph $H \in \mathcal{U}_{n,3}$, e.i. H is unicyclic with the cycle C_3 and for each edge $e \in E(H)$ the edge density γ_e is given. We extend some results presented in [5], where authors studied the inhomogeneous case for trees and proved the following theorem.

Theorem 6. *(Csikvári, Nagy [5]) Let T be a tree of order n and let v be a leaf of T. Assume that for each edge of T a density $\gamma_e = 1-r_e$ is given. Let T' be a tree obtained from T by deleting the leaf v and the edge uv, where u is the unique neighbour of v. Let the edge densities γ'_e in T' be defined as follows*

$$\gamma'_e=\begin{cases}\gamma_e=1-r_e, & \text{if } e \text{ is not incident to } u,\\ 1-\frac{r_e}{1-r_{uv}}, & \text{if } e \text{ is incident to } u.\end{cases}$$

Then the set of densities $\{\gamma_e\}_{e\in E(T)}$ ensures the existence of the factor T if and only if all $\gamma'_e \in (0,1]$ and the set of densities $\{\gamma'_e\}_{e\in E(T')}$ ensures the existence of the factor T'.

Theorem 6 provides authors of [5] with an efficient algorithm to decide whether a given set of edge densities in tree ensures the existence of a transversal or does not ensure. Their algorithm is presented below as **Algorithm** T for the completeness of our paper.

We extend the algorithm (**Algorithm** T) to the family of unicyclic graphs with the cycle C_3. The new algorithm (**Algorithm** $\mathcal{U}_{n,3}$) is based on the following Theorem 7 proved below by an extension of the method discovered in [5].

Theorem 7. *Let $H \in \mathcal{U}_{n,3}$ be a unicyclic graph of order n with the cycle C_3 and assume that for each edge $e \in E(H)$ a density $\gamma_e = 1-r_e$ is given. If the order of H is greater then 3, let v be a leaf of H and u be the unique neighbour of v, then let H' be a graph obtained from H by deleting the leaf v and an edge uv. Let the densities γ'_e in H' be defined as follows*

$$\gamma'_e=\begin{cases}\gamma_e=1-r_e, & \text{if } e \text{ is not incident to } u,\\ 1-\frac{r_e}{1-r_{uv}}, & \text{if } e \text{ is incident to } u.\end{cases}$$

If the order of H is equal to 3 (i.e., H is isomorphic to C_3 with $V(H)=\{a,b,c\}$), then let H' be a graph obtained from H by deleting the vertex a and edges ab and ac. H' is a path P_{bc}. Let the density γ'_{bc} in H' be defined as follows

$$\gamma'_{bc}=1-\frac{r_{bc}}{(1-r_{ab})(1-r_{ac})}.$$

Algorithm T

Step 0.
Let there be given a tree T^0 and edge densities γ_e^0. Set $T := T^0$ and $r_e = 1 - \gamma_e^0$.

Step 1.
Consider (T, r_e).

- **if** $|V(T)| = 2$ *and* $0 \le r_e < 1$ **then**

  STOP: the densities γ_e^0 ensure the existence of a factor T^0.

- **if** $|V(T)| \ge 2$ *and there exists an edge for which* $r_e \ge 1$ **then**

  STOP: the densities γ_e^0 do not ensure the existence of a factor T^0.

Step 2.
if $|V(T)| \ge 3$ *and* $0 \le r_e < 1$ *for all edges* $e \in E(T)$ **then**

DO pick a vertex v of degree 1 and let u be its unique neighbour. Let $T^{'} := T - v$ and

$$r_e^{'} = \begin{cases} r_e, & \text{if } e \text{ is not incident to } u,\\ \frac{r_e}{1-r_{uv}}, & \text{if } e \text{ is incident to } u. \end{cases}$$

Jump to *Step 1* with $(T, r_e) := (T^{'}, r_e^{'})$.

Then the set of densities $\{\gamma_e\}_{e\in E(H)}$ *ensures the existence of the factor* H *if and only if all* $\gamma_e^{'} \in (0,1]$ *and the set of densities* $\{\gamma_e^{'}\}_{e\in E(H')}$ *ensures the existence of the factor* $H^{'}$.

Proof. Let $H \in \mathcal{U}_{n,3}$ and let the set of densities $\gamma_e = 1 - r_e$ be given for each $e \in E(H)$. First we prove the following statement: if all $\gamma_e^{'}$ are indeed densities and they ensure the existence of a factor $H^{'}$, then the original densities γ_e ensure the existence of a factor H.

Let $G[H]$ be a blow-up of the graph H such that the density between A_i and A_j is at least γ_{ij}, where A_i is a cluster of the vertex $i \in V(H)$. We show that it contains a factor H.

Let us consider a graph $H \in \mathcal{U}_{n,3}$ with $n > 3$ vertices.

Let $v, u \in V(H)$, where v is a leaf of H and $u \in N_H(v)$.

Define $R_{v,u}$ as the subset of A_u in the following way (see Fig. 6).

$$R_{v,u} = \{x \in A_u \mid x \text{ is incident to some edge between } A_u \text{ and } A_v\}.$$

Note that

$$|R_{v,u}||A_v| \ge e(R_{v,u}, A_v) = \gamma_{uv}|A_u||A_v|.$$

Hence

$$|R_{v,u}| \ge \gamma_{uv}|A_u|.$$

Now we show the lower bound for the number of edges incident to $R_{v,u}$. Let $k \in N_H(u)$. By the inclusion - exclusion formula we count the lower bound for the number of edges between $R_{v,u}$ and A_k as follows.

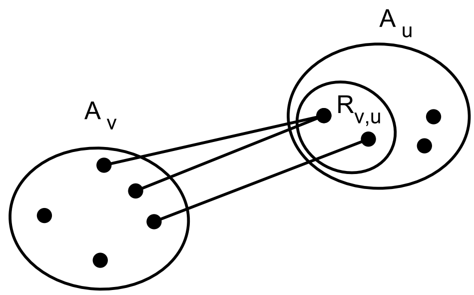

Fig. 6. Clusters A_v and A_u with the set $R_{v,u}$.

$$e(R_{v,u}, A_k) \ge e(A_u, A_k) - (|A_u| - |R_{v,u}|)|A_k| =$$
$$|R_{v,u}||A_k| + (\gamma_{ku} - 1)|A_k||A_u| \ge$$
$$|R_{v,u}||A_k| + (\gamma_{ku} - 1)\frac{1}{\gamma_{uv}}|R_{v,u}||A_k| =$$
$$\left(1 - \frac{r_{ku}}{1 - r_{uv}}\right)|R_{v,u}||A_k| = \gamma'_{ku}|R_{v,u}||A_k|.$$

Now, by deleting the vertex set A_v and $A_u \backslash R_{v,u}$ from $G[H]$, we obtain a graph which is a blow-up of $H^{'}$ with edge densities ensuring the existence of the factor $H^{'}$.

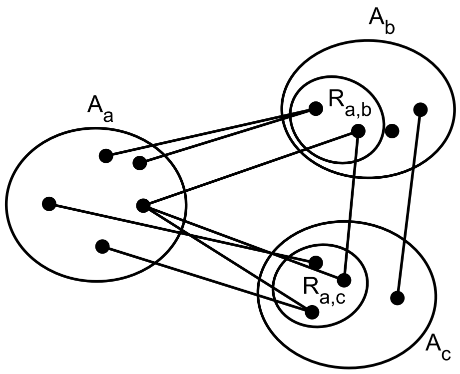

Fig. 7. Clusters A_a, A_b and A_c with the sets $R_{a,b}$ and $R_{a,c}$.

Moreover, by the definition of $R_{v,u}$ the factor $H^{'}$ can be extended to a factor H.

Now consider the situation when $n = 3$ and graph H is a cycle C_3 with the vertex set $\{a, b, c\}$. Let A_a be a cluster of vertex a. Define sets $R_{a,b}$ and $R_{a,c}$ in the following way (see Fig. 7).

$$R_{a,b} = \{x \in A_b \mid x \text{ is incident to some edge between } A_b \text{ and } A_a\},$$
$$R_{a,c} = \{x \in A_c \mid x \text{ is incident to some edge between } A_c \text{ and } A_a\}.$$

Note that

$$|R_b||A_a| \geq e(R_b, A_a) = \gamma_{ab}|A_a||A_b|,$$
$$|R_c||A_a| \geq e(R_c, A_a) = \gamma_{ac}|A_a||A_c|.$$

Hence we have the following lower bounds for the cardinalities of $R_{a,b}$ and $R_{a,c}$

$$|R_{a,b}| \geq \gamma_{ab}|A_b|$$

and

$$|R_{a,c}| \geq \gamma_{ac}|A_c|.$$

Next we show how many edges are incident to $R_{a,b}$ and $R_{a,c}$. Using the inclusion - exclusion formula we count the lower bound for the number of edges between $R_{a,b}$ and $R_{a,c}$

$$e(R_{a,b}, R_{a,c}) \geq e(A_b, A_a) - (|A_b| - |R_{a,b}|)|A_c| -$$
$$(|A_c| - |R_{a,c}|)|A_b| + (|A_b| - |R_{a,b}|)(|A_c| - |R_{a,c}|) =$$
$$|R_{a,b}||R_{a,c}| + (\gamma_{bc} - 1)|A_b||A_a| \geq$$
$$|R_{a,b}||R_{a,c}| + (\gamma_{bc} - 1)\frac{1}{\gamma_{ab}}\frac{1}{\gamma_{ac}}|R_{a,b}||R_{a,c}| =$$
$$\left(1 - \frac{r_{bc}}{(1 - r_{ab})(1 - r_{ac})}\right)|R_{a,b}||R_{a,c}| = \gamma^{'}_{bc}|R_{a,b}||R_{a,c}|.$$

Now, by deleting the vertex sets A_a, $A_b \backslash R_{a,b}$ and $A_c \backslash R_{a,c}$ from $G[C_3]$, we obtain a graph which is a blow-up of $C^{'}_3 = P_2$, $V(P_2) = \{b, c\}$, with edge densities ensuring the existence of the factor P_2. Moreover, by the definition of $R_{a,b}$ and $R_{a,c}$ the factor P_2 can be extended to a factor C_3.

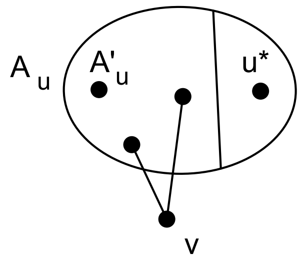

Fig. 8. We assume that $G^{'}[H^{'}]$ is without transversal $H^{'}$. The construction of the blow-up graph $G[H]$ without transversal H for the case where v is a leaf in H and $H^{'} = H - v$. The cluster $A^{'}_u$ is in $G^{'}[H^{'}]$. Let $A_u = A^{'}_u \cup u^*$ and $A_v = \{v\}$ be clusters in $G[H]$.

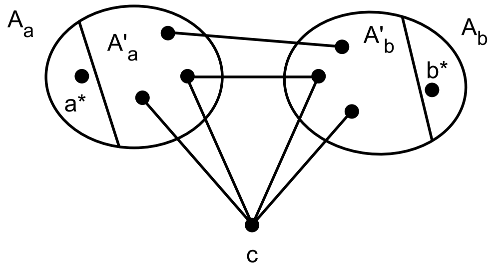

Fig. 9. We assume that $G^{'}[H^{'}]$ is without transversal $H^{'}$. The construction of the blow-up graph $G[H]$ without transversal H for the case where c is a vertex of C_3, $V(C_3) = \{a, b, c\}$ in H and $H^{'} = H - c$. The clusters $A^{'}_a$ and $A^{'}_b$ are in $G^{'}[H^{'}]$. Let $A_a = \{a^*\} \cup A^{'}_a$, $A_b = \{b^*\} \cup A^{'}_b$ and $A_c = \{c\}$ be clusters in $G[H]$.

Note that if

$$\gamma^{'}_{ku} < 0,$$

then

$$\gamma_{ku} + \gamma_{uv} < 1.$$

So there exists a construction which does not induce the path P_3 with the consecutive vertices k, u, v, where $i \in A_i$ ($i \in \{k, u, v\}$) in this case. Therefore, if some $\gamma^{'}_{ku} < 0$ then there exists a construction for a blow-up graph of H without a factor of H.

Next assume that all the $\gamma^{'}_e$ are proper densities, but there is a construction of a blow-up graph, say $G^{'}[H^{'}]$, with edge densities at least $\gamma^{'}_e$, but which does not induce a factor $H^{'}$. Thus we construct a blow-up $G[H]$ of the graph H not inducing H. We consider two possible cases. First let the picked vertex v be a leaf in H and $H^{'} = H - v$. Then set $A_u = \{u^*\} \cup A^{'}_u$ and $A_v = \{v\}$. We connect v to all elements of $A^{'}_u$ but do not connect to u^* without changing densities in $G^{'}[H^{'}]$ and with density γ_{vu} (see Fig. 8).

Now let $H^{'} = H - c$, where c is a vertex of C_3, $V(C_3) = \{a, b, c\}$. Then set $A_a = \{a^*\} \cup A^{'}_a$, $A_b = \{b^*\} \cup A^{'}_b$ and $A_c = \{c\}$. We connect c to all elements of $A^{'}_a$ and $A^{'}_b$ but do not connect to a^* and b^* without changing densities in $G^{'}[H^{'}]$ and with densities γ_{ca} and γ_{cb} (see Fig. 9). □

Theorem 7 provides us with the algorithm (**Algorithm** $\mathcal{U}_{n,3}$) to decide whether a given set of edge densities ensures the existence of a transversal H in a blow-up graph $G[H]$ or does not ensure, where $H \in \mathcal{U}_{n,3}$.

For further considerations recall some results presented in papers [3] and [5]. First lemma gives condition on edge densities in the triangle C_3 which allows us to check if these densities ensure existing of C_3 in a blow-up graph $G[C_3]$. The second results gives condition on existing graph H as a factor in a blow-up graph $G[H]$ in terms of the multivariate matching polynomial F_H.

Lemma 1. *(Bondy, et al. [3]) Let α, β, γ be the edge densities between the clusters of a blow-up graph of the triangle - a cycle C_3. If*

$$\alpha\beta + \gamma > 1, \beta\gamma + \alpha > 1, \gamma\alpha + \beta > 1,$$

then the blow-up graph contains a triangle as a transversal.

Theorem 8. *(Csikvári, Nagy [5]) Assume that for the graph H we have*

$$F_H(\underline{r_e}, t) > 0$$

for all $t \in [0,1]$ and some vector $\underline{r_e}$ of weights, where $r_e \in [0,1]$ for each edge $e \in E(H)$. Then the densities $\gamma_e = 1 - r_e$ ensure the existence H as a transversal.

Let $H := C_3$ with vertices a, b, c and edge densities $\gamma_e = 1 - r_e$, where $e \in \{ab, bc, ac\}$. Assume that all $r_e \in [0,1)$ and run the **Algorithm** $\mathcal{U}_{n,3}$ by deleting vertex a from the graph C_3 with edges incident to it (means ab and ac). As a result we get a graph $H^{'}$ as a path $P_2 = bc$ with edge density

$$\gamma'_{bc} = 1 - r'_{bc} = 1 - \frac{r_{bc}}{(1 - r_{ab})(1 - r_{ac})}.$$

For $H^{'}$ we have

$$F_{H'}(\underline{r_e}, t) = 1 - tr'_{bc}.$$

By Theorem 8 we need

$$F_{H'}(\underline{r_e}, t) > 0$$

for $t \in [0,1]$.

Hence

$$\frac{1}{r'_{bc}} > 1,$$

$$(1 - r_{ab})(1 - r_{ac}) - r_{bc} > 0$$

and

$$\gamma_{ab}\gamma_{ac} + \gamma_{bc} > 1.$$

Similar inequalities are received when, instead of a vertex a, we delete in **Algorithm** $\mathcal{U}_{n,3}$ vertex b or c. As we can see we have a result presented in Proposition 1 consensual with Lemma 1.

Proposition 1. *Let a, b, c be vertices in a triangle C_3. Assume that $\gamma_e = 1 - r_e$ be an edge density assigned to each edge $e \in E(C_3)$, where $E(C_3) = \{ab, ac, bc\}$. If*

$$\frac{r_{ab}}{(1 - r_{ac})(1 - r_{bc})} < 1, \ \frac{r_{ac}}{(1 - r_{ab})(1 - r_{bc})} < 1$$

$$\textit{and } \frac{r_{bc}}{(1 - r_{ab})(1 - r_{ac})} < 1,$$

then the set of densities $\{\gamma_e\}_{e \in E(C_3)}$ ensures existence of a transversal C_3 in a blow-up graph $G[C_3]$.

By running **Algorithm** $\mathcal{U}_{n,3}$ on some unicyclic graph $H \in \mathcal{U}_{n,3}$ with $\gamma_e = 1 - tr_e$ and using the multivariate matching polynomial $F_H(\underline{r_e}, s)$ we can prove the following lemma.

Lemma 2. *Let H be a weighted unicyclic graph of order $n > 2$ with the cycle C_3. Let $\gamma_e = 1 - tr_e$ be densities assigned to each edge $e \in E(H)$, where $r_e \in [0,1)$. Assume that after running Algorithm $\mathcal{U}_{n,3}$ we get a cycle C_3 with*

$$F_{C_3}(\underline{r_e}, t) = 0,$$

then t is a root of the multivariate matching polynomial $F_H(\underline{r_e}, s)$ of the graph H.

Proposition 2. *Let H be a weighted unicyclic graph of order $n > 2$ with the cycle C_3. Let $\gamma_e = 1 - tr_e$ be the densities assigned to each edge $e \in E(H)$. Assume that after running Algorithm $\mathcal{U}_{n,3}$ we get a cycle C_3 with the vertex set $V(C_3) = \{a, b, c\}$ and with $F_{C_3}(\underline{r_e}, t) = 0$ and, after restart Algorithm $\mathcal{U}_{n,3}$, we get a path P_2 (by deleting the vertex a and edges ab, ac), then*

$$F_{P_2}(\underline{r_e^{'}}, s) = \frac{t^2 r_{ab} r_{ac} + tr_{bc}}{(1 - tr_{ab})(1 - tr_{ac})} - s\frac{tr_{bc}}{(1 - tr_{ab})(1 - tr_{ac})}.$$

Proof. Assume that after running **Algorithm** $\mathcal{U}_{n,3}$ we get a cycle C_3 with edge densities $\gamma_e = 1 - tr_e$. Let $V(C_3) = \{a, b, c\}$ and $r_{ab}, r_{ac}, r_{bc} \in [0,1)$. The multivariate matching polynomial

$$F_{C_3}(\underline{r_e}, s) = 1 - s(r_{ab} + r_{ac} + r_{bc})$$

has exactly one root

$$t = \frac{1}{(r_{ab} + r_{ac} + r_{bc})}.$$

By deleting vertex a from the cycle C_3 with the edges $e_{ab} = ab$ and $e_{ac} = ac$ we obtain a path $P_2 = bc$. By Theorem 7 we get that

$$F_{P_2}(\underline{r_e^{'}}, s) = 1 - sr_{bc}^{'} = 1 - s\frac{tr_{bc}}{(1 - tr_{ab})(1 - tr_{ac})}.$$

By multiplying both sides by

$$(1 - tr_{ab})(1 - tr_{ac})$$

we have

$$(1 - tr_{ab})(1 - tr_{ac})F_{P_2}(\underline{r_e^{'}}, s) =$$

$$(1 - tr_{ab})(1 - tr_{ac}) - str_{bc} =$$

$$1 - t(r_{ab} + r_{ac} + r_{bc}) + tr_bc + t^2 r_{ab} r_{ac} - str_{bc}.$$

So

$$(1 - tr_{ab})(1 - tr_{ac})F_{P_2}(\underline{r_e^{'}}, s) - t^2 r_{ab} r_{ac} - tr_{bc} + str_{bc} =$$

$$F_{C_3}(\underline{r_e}, t) = 0.$$

Algorithm $U_{n,3}$

Input: a unicyclic graph $H \in \mathcal{U}_{n,3}$ with the set of edge densities $\{\gamma_e\}_{e \in E(H)}$.

Output: a boolean value

$$D = \begin{cases} TRUE, & \text{the densities } \gamma_e \text{ ensure the existence of a factor } H, \\ FALSE, & \text{the densities } \gamma_e \text{ does not ensure the existence of a factor } H. \end{cases}$$

Consider a weighted graph (H, r_e), where $r_e = 1 - \gamma_e$.

Step 1.

- **if** $|V(H)| = 2$ *(means H is a path P_2) and* $0 \leq r_e < 1$ **then**

  STOP: D :=TRUE.

- **if** $|V(H)| \geq 2$ *and there exists an edge for which* $r_e \geq 1$ **then**

  STOP: D :=FALSE.

Step 2.

- **if** $|V(H)| = 3$ *(means H is a cycle C_3) and* $0 \leq r_e < 1$ *for all edges* $e \in E(H)$ **then**

  pick a vertex c of the graph H and let a, b be its neighbours. Let $H^{'} := H - c$ and

$$r^{'}_{ab} = \frac{r_{ab}}{(1 - r_{ac})(1 - r_{bc})}.$$

- **if** $|V(H)| > 3$ *and* $0 \leq r_e < 1$ *for all edges* $e \in E(H)$ **then**

  pick a vertex v of degree 1 and let u be its unique neighbour. Let $H^{'} := H - v$ and

$$r^{'}_e = \begin{cases} r_e, & \text{if } e \text{ is not incident to } u, \\ \frac{r_e}{1 - r_{uv}}, & \text{if } e \text{ is incident to } u. \end{cases}$$

Go to *Step 1* with $(H, r_e) := (H^{'}, r^{'}_e)$.

Hence

$$F_{P_2}(\underline{r^{'}_e}, s) = \frac{t^2 r_{ab} r_{ac} + t r_{bc}}{(1 - t r_{ab})(1 - t r_{ac})} - s \frac{t r_{bc}}{(1 - t r_{ab})(1 - t r_{ac})}.$$

By the definition of $F_{P_2}(\underline{r^{'}_e}, s)$ we have

$$\frac{t^2 r_{ab} r_{ac} + t r_{bc}}{(1 - t r_{ab})(1 - t r_{ac})} = 1$$

and

$$t(r_{ac} + r_{ab} + r_{bc}) = 1.$$

□

Note that if

$$\gamma^{'}_{bc} = 1 - \frac{t r_{bc}}{(1 - t r_{ab})(1 - t r_{ac})} = 0,$$

then

$$t r_{bc} = (1 - t r_{ab})(1 - t r_{ac})$$

and

$$\frac{t^2 r_{ab} r_{ac}}{(1 - t r_{ab})(1 - t r_{ac})} = 0,$$

$$t r_{ab} t r_{ac} = 0.$$

Therefore,

$$t(r_{ac} + r_{ab} + r_{bc}) = 1 + t r_{ab} t r_{ac} = 1.$$

So t is the root of $F_{C_3}(\underline{r_e}, t)$.

From above consideration we deduce that **Algorithm** $\mathcal{U}_{n,3}$ works correctly with time complexity at most $\mathcal{O}(n^2)$. **Algorithm** $\mathcal{U}_{n,3}$ can be implemented in such a way that a vertex of the subgraph C_3 be considered (picked) in the last step of the algorithm.

IV. Conclusion

We have presented **Algorithm** $\mathcal{U}_{n,3}$ for testing whether the unicyclic graph $H \in \mathcal{U}_{n,3}$ with the set of edge densities $\{\gamma_e\}_{e\in E(H)}$ is a factor of a blow-up graph $G[H]$. Precisely, we have the answer whether the edge densities ensure the existence of the factor or do not ensure. In future work we will study the density Turán problem for an arbitrary graph of the family $\mathcal{U}_{n,p}$, $p \geq 4$, and for other families of graphs. Moreover, we wish to construct efficient algorithms for testing the existence of blow-up graphs with factors of the families.

Open problem: Look for the density Turán problem algorithm for families of connected graphs with blocks (i.e., 2-connected components) isomorphic to cycles and/or P_2 .

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