

# Symbolic description of the polynomial roots and their numerical implementation – better than in Mathematica software?

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**Abstract**—This paper is a continuation of the discussion undertaken in paper [31]. We present in the current paper the corrected, and also given in a slightly changed form, Vandermonde formulae for the roots of some quintic polynomials considered in J.P. Tignol’s monograph [26]. The proofs of selected trigonometric identities from [31] are given and some new identities have been generated by the occasion, which also can be used for testing our Lagrange algorithm for the case of cubic polynomials. Moreover, we present here the decomposition of polynomials belonging to some two-parameter family of polynomials related to the Chebyshev polynomials of the first kind.

**Keywords** – Lagrange algorithm, Chebyshev polynomials

## I. INTRODUCTION

In paper [31] we have presented an algorithm (the so called Lagrange algorithm) for determining the complex roots of real polynomials which appeared to be more effective than the symbolic calculations proposed in Mathematica software. Many cubic polynomial, considering the polynomials used for testing our Lagrange algorithm, led us to some attractive trigonometric relations (see the appendix which is an important supplement of paper [31]). In the current paper we propose to repeat our previous considerations for the case of quintic polynomials and for the polynomials of higher degrees as well.

Let us also recall that the complex roots of quintic polynomials and the polynomials of higher degree, described with the aid of elliptic functions and hypergeometric functions, are well known (see [2], [12]). Certainly, the computational usefulness of such solutions is not good, it still remains in the sphere of future discoveries. At present, we have to limit the calculations, especially the symbolic calculations, just for the selected families of polynomials, for which the formulae describing roots are well known and are in the “numerically accepted” form.

## II. VANDERMONDE FORMULAE

Next of the investigated quintic polynomials

$$x^5 - x^4 - 4x^3 + 3x^2 + 3x - 1 = \prod_{k=1}^5 \left( x + 2 \cos \frac{2k\pi}{11} \right)$$

delivered to us many unexpected emotions. So in monograph [26] the following identity, based on the Lagrange resolvent method of solving the algebraic equations, is suggested

$$x = \frac{1}{5} (1 + \Delta' + \Delta'' + \Delta''' + \Delta^{IV}), \quad (1)$$

where  $x$  is one of the roots of this polynomial. However the description of expressions  $\Delta'$ ,  $\Delta''$ ,  $\Delta'''$  and  $\Delta^{IV}$ , given in [26], “seems to be” incorrect. It should be, and this is already our correction of these relations:  $\Delta' = \sqrt[5]{d_1}$ ,  $\Delta'' = \overline{\Delta'}$ ,  $\Delta''' = \sqrt[5]{d_3}$ ,  $\Delta^{IV} = \overline{\Delta'''}$ , more precisely,  $\Delta'$  and  $\Delta'''$  are any values of the respective complex roots of degree five taken from numbers (respectively):

$$d_1 = \frac{11}{4} \left( 89 + 25\sqrt{5} - 5i\sqrt{5 + 2\sqrt{5}} + 45i\sqrt{5 - 2\sqrt{5}} \right),$$

$$d_3 = \frac{11}{4} \left( 89 - 25\sqrt{5} - 45i\sqrt{5 + 2\sqrt{5}} - 5i\sqrt{5 - 2\sqrt{5}} \right).$$

Identity (1) takes then the following trigonometric form<sup>1</sup> (the calculations were executed by hand with the intensive support

<sup>1</sup>We used in calculations the following identity

$$d\sqrt{a + b\sqrt{a}} + \sqrt{a - b\sqrt{a}} = \sqrt{(d^2 + 1)a + (b(d^2 - 1) + 2cd)\sqrt{a}},$$

where  $a = b^2 + c^2$ ,  $b, c \in \mathbb{C}$  (the complex cases need some additional verification).

of Mathematica software):

$$-2 \cos \frac{2k\pi}{11} = \frac{1}{5} \left( 1 + 2\sqrt{11} \cos \left( \frac{1}{5} \left( t_i + s_i \arctan \frac{\Im d_1}{\Re d_1} \right) \right) \right. \\ \left. + \sqrt{11} \cos \left( \frac{1}{5} \left( t_j - r_j \arctan \frac{\Im d_3}{\Re d_3} \right) \right) \right)$$

and hence, after simple algebraic transformations, we get

$$-5 \cos \frac{2k\pi}{11} - \frac{1}{2} = \sqrt{11} \left( \cos \left( \frac{1}{5} \left( t_i + s_i \arctan \frac{p}{q} \right) \right) \right. \\ \left. + \cos \left( \frac{1}{5} \left( t_j + r_j \arctan \frac{pq}{109} \right) \right) \right), \quad (2)$$

where  $p = 5\sqrt[4]{5}$ ,  $q = \sqrt{877\sqrt{5} + 1958}$ ,  $s_i, r_i$  are the signs and  $t_n := 2(n-1)\pi$  for every  $n = 1, 2, \dots, 5$ . By using the Mathematica software we have found the following sequences of the values of indices  $k, i, j$  and the signs  $s_i, r_j$  for which both the above identities hold true:

$k$	$i$	$j$	$s_i$	$r_j$	$k$	$i$	$j$	$s_i$	$r_j$
1	3	3	-1	-1	3	1	4	+1	-1
1	3	4	-1	+1	3	1	4	-1	-1
1	4	3	+1	-1	4	2	2	+1	-1
1	4	4	+1	+1	4	2	5	+1	+1
2	3	2	+1	+1	4	5	2	-1	-1
2	3	5	+1	-1	4	5	5	-1	+1
2	4	2	-1	+1	5	2	1	-1	-1
2	4	5	-1	-1	5	2	1	-1	+1
3	1	3	+1	+1	5	5	1	+1	-1
3	1	3	-1	+1	5	5	1	+1	+1

### III. QUINTIC POLYNOMIALS

In this section we discuss the monic irreducible quintic polynomials  $f \in \mathbb{Q}[x]$ . If the equation  $f(x) = 0$  is solvable by radicals, the quintic polynomial  $f(x)$  is said to be solvable. First let us consider the quintic polynomial in the reduced form

$$x^5 + a_2x^2 + a_1x + a_0 = 0,$$

called the principal quintic form, and

$$x^5 + b_1x + b_0 = 0,$$

called the Bring-Jerrard quintic form (see [1]).

It is known that if  $f(x) = x^5 + ax^i + b \in \mathbb{Q}^*[x]$ ,  $i = 1, 2$ , where  $\mathbb{Q}^* := \mathbb{Q} \setminus \{0\}$ , and  $\text{disc } f(x) > 0^2$ , then  $f(x)$  has exactly one real root [24].

B.K. Spearman and K.S. Williams proved in [23] the following result.

**Theorem 1.** *Let  $a, b \in \mathbb{Q}^*$  be such that the quintic trinomial  $f(x) = x^5 + ax + b$  is irreducible. Then  $f(x)$  is solvable if and only if there exist the rational numbers  $\varepsilon \in \{-1, 1\}$ ,  $c \geq 0$  and  $\alpha \neq 0$  such that*

$$a = \frac{5\alpha^4(3 - 4\varepsilon c)}{c^2 + 1}, \quad b = \frac{-4\alpha^5(11\varepsilon + 2c)}{c^2 + 1}.$$

Moreover, if we set

$$D = c^2 + 1,$$

<sup>2</sup>The discriminant of  $f$ .

$$\begin{cases} v_1 = \sqrt{D} + \sqrt{D - \varepsilon\sqrt{D}}, & v_2 = -\sqrt{D} - \sqrt{D + \varepsilon\sqrt{D}}, \\ v_3 = -\sqrt{D} + \sqrt{D + \varepsilon\sqrt{D}}, & v_4 = \sqrt{D} - \sqrt{D - \varepsilon\sqrt{D}}, \end{cases}$$

$$u_1 = \sqrt[5]{\frac{v_1^2 v_3}{D^2}}, \quad u_2 = \sqrt[5]{\frac{v_3^2 v_4}{D^2}}, \quad u_3 = \sqrt[5]{\frac{v_2^2 v_1}{D^2}}, \quad u_4 = \sqrt[5]{\frac{v_4^2 v_2}{D^2}},$$

and  $w = \exp(2\pi i/5)$ , then the roots of  $x^5 + ax + b$  are equal to  $x_0, x_1, x_2, x_3, x_4$ , where

$$\frac{x_i}{\alpha} = \sum_{k=1}^4 w^{ki} u_k$$

$$= w^i u_1 + \frac{w^{2i} u_1^2 \sqrt{D} \sqrt[5]{\varepsilon}}{v_1} + \frac{w^{3i} u_1^3 D}{v_1 v_3} - \frac{w^{4i} u_1^4 \sqrt{D^3}}{v_1^2 v_3} \sqrt[5]{\frac{\varepsilon}{D}},$$

for every  $i = 0, 1, 2, 3, 4$ .

For illustrating the application of the formulae from the above theorem let us find the complex roots of polynomial

$$x^5 + 11x + 44,$$

which we get for parameters  $\varepsilon = 1$ ,  $\alpha = -1$ ,  $c = 2/11$ . By using the above formulae, supported by Mathematica software, we derive the following formulae

$$x_0 = \lambda(\xi_1 - \xi_2 - \xi_3 - \xi_4) \approx -1.87775,$$

$$x_1 = -\lambda \left( e^{\frac{3\pi i}{5}} \xi_1 + e^{\frac{2\pi i}{5}} \xi_2 + e^{\frac{4\pi i}{5}} \xi_3 - e^{\frac{\pi i}{5}} \xi_4 \right)$$

$$\approx 1.80012 - 1.44746i,$$

$$x_2 = -\lambda \left( e^{\frac{\pi i}{5}} \xi_1 + e^{\frac{4\pi i}{5}} \xi_2 - e^{\frac{3\pi i}{5}} \xi_3 + e^{\frac{2\pi i}{5}} \xi_4 \right)$$

$$\approx -0.861241 - 1.9105i,$$

$$x_3 = \bar{x}_2, \quad x_4 = \bar{x}_1,$$

where  $\lambda = -\sqrt[5]{11/5^4}$  and

$$\xi_k := \left( (-1)^{k+1} (50\sqrt{5} - 75) + \sqrt{18125 - 6169\sqrt{5}} \right)^{1/5},$$

$$\xi_l := \left( 50\sqrt{5} + 75 + (-1)^l \sqrt{18125 + 6169\sqrt{5}} \right)^{1/5},$$

for  $k = 1, 2$  and  $l = 3, 4$ .

J.A. Johnstone and B.K. Spearman in paper [11] proved the following result.

**Theorem 2.** *Let  $f(x) = x^5 + x - a$ , where  $a \in \mathbb{Z}$ . Then  $f(x)$  is not solvable by radicals unless  $a = r^5 + r$  for some integer  $r$ , or  $a = \pm 1, \pm 6$ . The last case was discussed and proved earlier by S. Rabinowitz [21].*

B.K. Spearman and K.S. Williams in paper [24] proved also the completely unexpected result, given below.

**Theorem 3.** *There exist only five essentially different, irreducible, solvable, quintic trinomials  $x^5 + ax^2 + b$ ,  $a, b \in \mathbb{Q}^*$ , namely:  $x^5 + 5x^2 + 3$ ,  $x^5 + 5x^2 - 15$ ,  $x^5 + 25x^2 + 300$ ,  $x^5 + 100x^2 + 1000$  and  $x^5 + 250x^2 + 625$ .*

## IV. APPENDIX

In paper [31] we have presented the following formulae

$$\begin{aligned} & \left(z - 2 \sin \frac{4\pi}{13}\right) \left(z - 2 \sin \frac{10\pi}{13}\right) \left(z - 2 \sin \frac{12\pi}{13}\right) \\ &= z^3 - \sqrt{\frac{13+3\sqrt{13}}{2}} z^2 + \sqrt{13} z - \sqrt{\frac{13-3\sqrt{13}}{2}} \end{aligned} \quad (3)$$

$$\begin{aligned} & \left(z - 2 \sin \frac{2\pi}{13}\right) \left(z - 2 \sin \frac{6\pi}{13}\right) \left(z + 2 \sin \frac{8\pi}{13}\right) \\ &= z^3 - \sqrt{\frac{13-3\sqrt{13}}{2}} z^2 - \sqrt{13} z + \sqrt{\frac{13+3\sqrt{13}}{2}} \end{aligned} \quad (4)$$

with the suggestion that they can be derived on the basis of formulae given below

$$\begin{aligned} & z^3 + \frac{1-\sqrt{13}}{2} z^2 - z + \frac{\sqrt{13}+3}{2} \\ &= \left(z - 2 \cos \frac{2\pi}{13}\right) \left(z - 2 \cos \frac{6\pi}{13}\right) \left(z - 2 \cos \frac{8\pi}{13}\right), \end{aligned} \quad (5)$$

$$\begin{aligned} & z^3 + \frac{1+\sqrt{13}}{2} z^2 - z - \frac{\sqrt{13}+3}{2} \\ &= \left(z - 2 \cos \frac{4\pi}{13}\right) \left(z - 2 \cos \frac{10\pi}{13}\right) \left(z - 2 \cos \frac{12\pi}{13}\right). \end{aligned} \quad (6)$$

To explain it better we decided to "reveal" here the details of the proof. Thus, by proving decomposition (3) and (4) we used the following relations

$$\begin{aligned} & 8 \sin \frac{4\pi}{13} \sin \frac{10\pi}{13} \sin \frac{12\pi}{13} = i \left( e^{i\frac{4\pi}{13}} - e^{-i\frac{4\pi}{13}} \right) \\ & \quad \times \left( e^{i\frac{10\pi}{13}} - e^{-i\frac{10\pi}{13}} \right) \left( e^{i\frac{12\pi}{13}} - e^{-i\frac{12\pi}{13}} \right) \\ &= 2 \sin \frac{2\pi}{13} + 2 \sin \frac{6\pi}{13} - 2 \sin \frac{8\pi}{13}, \end{aligned}$$

$$\begin{aligned} & \left( 2 \sin \frac{4\pi}{13} + 2 \sin \frac{10\pi}{13} + 2 \sin \frac{12\pi}{13} \right)^2 \\ &= 6 + 2 \left( \cos \frac{2\pi}{13} + \cos \frac{6\pi}{13} + \cos \frac{8\pi}{13} \right) \\ & \quad - 4 \left( \cos \frac{4\pi}{13} + \cos \frac{10\pi}{13} + \cos \frac{12\pi}{13} \right) \stackrel{(5)}{(6)} \frac{13+3\sqrt{13}}{2}, \end{aligned}$$

$$\begin{aligned} & 4 \left( \sin \frac{4\pi}{13} \sin \frac{10\pi}{13} + \sin \frac{4\pi}{13} \sin \frac{12\pi}{13} + \sin \frac{10\pi}{13} \sin \frac{12\pi}{13} \right) = \\ &= 2 \left( \cos \frac{2\pi}{13} + \cos \frac{6\pi}{13} + \cos \frac{8\pi}{13} \right) \\ & \quad - 2 \left( \cos \frac{4\pi}{13} + \cos \frac{10\pi}{13} + \cos \frac{12\pi}{13} \right) \stackrel{(5)}{(6)} \sqrt{13}, \end{aligned}$$

$$\begin{aligned} & 8 \sin \frac{2\pi}{13} \sin \frac{6\pi}{13} \sin \frac{8\pi}{13} \\ &= i \left( e^{i\frac{2\pi}{13}} - e^{-i\frac{2\pi}{13}} \right) \left( e^{i\frac{6\pi}{13}} - e^{-i\frac{6\pi}{13}} \right) \left( e^{i\frac{8\pi}{13}} - e^{-i\frac{8\pi}{13}} \right) \\ &= 2 \sin \frac{4\pi}{13} + 2 \sin \frac{10\pi}{13} + 2 \sin \frac{12\pi}{13}, \end{aligned}$$

$$\begin{aligned} & \left( 2 \sin \frac{2\pi}{13} + 2 \sin \frac{6\pi}{13} - 2 \sin \frac{8\pi}{13} \right)^2 \\ &= 6 + 2 \left( \cos \frac{4\pi}{13} + \cos \frac{10\pi}{13} + \cos \frac{12\pi}{13} \right) \\ & \quad - 4 \left( \cos \frac{2\pi}{13} + \cos \frac{6\pi}{13} - \cos \frac{8\pi}{13} \right) \stackrel{(5)}{(6)} \frac{13-3\sqrt{13}}{2}, \\ & 4 \left( \sin \frac{2\pi}{13} \sin \frac{6\pi}{13} - \sin \frac{2\pi}{13} \sin \frac{8\pi}{13} - \sin \frac{6\pi}{13} \sin \frac{8\pi}{13} \right) \\ &= 2 \left( \cos \frac{4\pi}{13} + \cos \frac{10\pi}{13} + \cos \frac{12\pi}{13} \right) \\ & \quad - 2 \left( \cos \frac{2\pi}{13} + \cos \frac{6\pi}{13} + \cos \frac{8\pi}{13} \right) \stackrel{(5)}{(6)} -\sqrt{13}. \end{aligned}$$

In order to complete the collection of formulae (3)–(6) we give additionally the decompositions connected with the values of tangent and cotangent functions of the respective ternary sets of arguments  $\left\{ \frac{2\pi}{13}, \frac{6\pi}{13}, \frac{8\pi}{13} \right\}$  and  $\left\{ \frac{4\pi}{13}, \frac{10\pi}{13}, \frac{12\pi}{13} \right\}$ . They are of the following form

$$\begin{aligned} & \left( x - \tan \frac{4\pi}{13} \right) \left( x - \tan \frac{10\pi}{13} \right) \left( x - \tan \frac{12\pi}{13} \right) \\ &= x^3 - \sqrt{65 - 18\sqrt{13}} x^2 + (13 - 4\sqrt{13}) x - \sqrt{65 - 18\sqrt{13}}, \end{aligned}$$

since from formulae (3)–(6) we easily obtain the equalities

$$\begin{aligned} & \tan \frac{4\pi}{13} \tan \frac{10\pi}{13} \tan \frac{12\pi}{13} = \frac{2 \sin \frac{4\pi}{13} 2 \sin \frac{10\pi}{13} 2 \sin \frac{12\pi}{13}}{2 \cos \frac{4\pi}{13} 2 \cos \frac{10\pi}{13} 2 \cos \frac{12\pi}{13}} \\ &= \frac{\sqrt{\frac{13-3\sqrt{13}}{2}}}{\frac{\sqrt{13+3}}{2}} = \frac{\sqrt{13}-3}{4} \sqrt{26-6\sqrt{13}} = \sqrt{65-18\sqrt{13}}, \\ & \tan \frac{4\pi}{13} + \tan \frac{10\pi}{13} + \tan \frac{12\pi}{13} = \tan \frac{4\pi}{13} \tan \frac{10\pi}{13} \tan \frac{12\pi}{13},^3 \end{aligned}$$

<sup>3</sup>The following identity holds

$$\tan x + \tan y + \tan(k\pi - x - y) = \tan x \tan y \tan(k\pi - x - y)$$

for the respective values of  $x, y \in \mathbb{R}$  and  $k \in \mathbb{Z}$ , which in turn implies the identity

$$\cot x \cot y + \cot x \cot(k\pi - x - y) + \cot y \cot(k\pi - x - y) = 1.$$

Moreover, if  $x_1 + x_2 + x_3 = 2\pi$ ,  $x_1, x_2, x_3 \in \mathbb{R} \setminus \pi\mathbb{Z}$ , then the following identities hold

$$\begin{aligned} & -2(\cot x_1 + \cot x_2 + \cot x_3) \sin x_1 \sin x_2 \sin x_3 \\ &= \sin^2 x_1 + \sin^2 x_2 + \sin^2 x_3 = (\sin x_1 + \sin x_2 + \sin x_3)^2 \\ & \quad - 2 \sin x_1 \sin x_2 - 2 \sin x_1 \sin x_3 - 2 \sin x_2 \sin x_3. \end{aligned}$$

*Proof.* We have

$$\begin{aligned} \cot x_1 + \cot x_2 &= -\frac{\sin^2 x_3}{\sin x_1 \sin x_2 \sin x_3}, \\ \cot x_1 + \cot x_3 &= -\frac{\sin^2 x_2}{\sin x_1 \sin x_2 \sin x_3}, \\ \cot x_2 + \cot x_3 &= -\frac{\sin^2 x_1}{\sin x_1 \sin x_2 \sin x_3}, \end{aligned}$$

from which, by summing respectively by sides, we obtain the expected equality.  $\square$

the last formula implies the following one

$$\begin{aligned} \cot \frac{4\pi}{13} + \cot \frac{10\pi}{13} + \cot \frac{12\pi}{13} &= \frac{\frac{1}{4} \frac{13+3\sqrt{13}}{2} - \frac{\sqrt{13}}{2}}{-\frac{1}{4} \sqrt{\frac{13-3\sqrt{13}}{2}}} \\ &= -\frac{\sqrt{2}}{4} \sqrt{13+3\sqrt{13}}(\sqrt{13}-1) = -\frac{\sqrt{2}}{2} \sqrt{26+4\sqrt{13}}, \\ \tan \frac{4\pi}{13} \tan \frac{10\pi}{13} + \tan \frac{4\pi}{13} \tan \frac{12\pi}{13} + \tan \frac{10\pi}{13} \tan \frac{12\pi}{13} \\ &= \tan \frac{4\pi}{13} \tan \frac{10\pi}{13} \tan \frac{12\pi}{13} \left( \cot \frac{4\pi}{13} + \cot \frac{10\pi}{13} + \cot \frac{12\pi}{13} \right) \\ &= \sqrt{65-18\sqrt{13}} \left( -\frac{\sqrt{2}}{2} \sqrt{26+4\sqrt{13}} \right) \\ &= -\sqrt{(65-18\sqrt{13})(13+2\sqrt{13})} = 13-4\sqrt{13}. \end{aligned}$$

The next formulae, derived by us on the way of algebraic tests of the Lagrange algorithm, are as follows (we present them without the proofs with respect to the length of this paper):

$$\begin{aligned} \left( x - \cot \frac{4\pi}{13} \right) \left( x - \cot \frac{10\pi}{13} \right) \left( x \cot \frac{12\pi}{13} \right) \\ = x^3 + \frac{\sqrt{2}}{2} \sqrt{26+4\sqrt{13}} x^2 + x - \sqrt{\frac{1}{13}(65+18\sqrt{13})}, \end{aligned}$$

$$\begin{aligned} \left( x - \tan \frac{2\pi}{13} \right) \left( x - \tan \frac{6\pi}{13} \right) \left( x + \tan \frac{8\pi}{13} \right) = x^3 - \\ \sqrt{65-18\sqrt{13}} x^2 + (13+4\sqrt{13})x - \sqrt{65-18\sqrt{13}}, \end{aligned}$$

$$\begin{aligned} \left( x - \cot \frac{2\pi}{13} \right) \left( x - \cot \frac{6\pi}{13} \right) \left( x + \cot \frac{8\pi}{13} \right) \\ = x^3 - \sqrt{13-2\sqrt{13}} x^2 + x - \sqrt{\frac{1}{13}(65-18\sqrt{13})}, \end{aligned}$$

$$\begin{aligned} \left( x - 2 \cos \frac{2\pi}{21} \right) \left( x - 2 \cos \frac{8\pi}{21} \right) \left( x - 2 \cos \frac{32\pi}{21} \right) \\ = \left( x - 2 \cos \frac{2\pi}{21} \right) \left( x - 2 \cos \frac{8\pi}{21} \right) \left( x + 2 \cos \frac{11\pi}{21} \right) \\ = x^3 - \frac{1+\sqrt{21}}{2} x^2 - \frac{1-\sqrt{21}}{2} x - \frac{5-\sqrt{21}}{2}, \end{aligned}$$

$$\begin{aligned} \left( x - 2 \cos \frac{4\pi}{21} \right) \left( x - 2 \cos \frac{16\pi}{21} \right) \left( x - 2 \cos \frac{64\pi}{21} \right) \\ = \left( x - 2 \cos \frac{4\pi}{21} \right) \left( x + 2 \cos \frac{5\pi}{21} \right) \left( x + 2 \cos \frac{\pi}{21} \right) \\ = x^3 - \frac{1-\sqrt{21}}{2} x^2 - \frac{1+\sqrt{21}}{2} x - \frac{5+\sqrt{21}}{2}, \end{aligned}$$

since  $\cos \frac{16\pi}{21} = -\cos \frac{5\pi}{21}$ ,  $\cos \frac{32\pi}{21} = \cos \frac{10\pi}{21} = -\cos \frac{11\pi}{21}$  and  $\cos \frac{64\pi}{21} = -\cos \frac{\pi}{21} = \cos \frac{20\pi}{21}$ . Finally the last formula is of

the form

$$\begin{aligned} \prod_{k=1}^6 \left( x - 2 \cos \frac{2^k \pi}{21} \right) = \\ = \left( x^3 - \frac{1+\sqrt{21}}{2} x^2 - \frac{1-\sqrt{21}}{2} x - \frac{5-\sqrt{21}}{2} \right) \\ \times \left( x^3 - \frac{1-\sqrt{21}}{2} x^2 - \frac{1+\sqrt{21}}{2} x - \frac{5+\sqrt{21}}{2} \right) \\ = x^6 - x^5 - 6x^4 + 6x^3 + 8x^2 - 8x + 1. \end{aligned}$$

Furthermore we have (see [29]):

$$\begin{aligned} \left( x - 2 \sin \frac{2\pi}{21} \right) \left( x - 2 \sin \frac{8\pi}{21} \right) \left( x - 2 \sin \frac{32\pi}{21} \right) \\ = \left( x - 2 \sin \frac{2\pi}{21} \right) \left( x - 2 \sin \frac{8\pi}{21} \right) \left( x + 2 \sin \frac{10\pi}{21} \right) \\ = x^3 + \frac{\sqrt{3}-\sqrt{7}}{2} x^2 - \frac{3+\sqrt{21}}{2} x + \frac{\sqrt{3}+\sqrt{7}}{2}, \end{aligned}$$

$$\begin{aligned} \left( x - 2 \sin \frac{4\pi}{21} \right) \left( x - 2 \sin \frac{16\pi}{21} \right) \left( x - 2 \sin \frac{64\pi}{21} \right) \\ = \left( x - 2 \sin \frac{4\pi}{21} \right) \left( x - 2 \sin \frac{5\pi}{21} \right) \left( x + 2 \sin \frac{\pi}{21} \right) \\ = x^3 - \frac{\sqrt{3}+\sqrt{7}}{2} x^2 + \frac{\sqrt{21}-3}{2} x + \frac{\sqrt{7}-\sqrt{3}}{2}, \end{aligned}$$

since  $\sin \frac{16\pi}{21} = \sin \frac{5\pi}{21}$ ,  $\sin \frac{32\pi}{21} = -\sin \frac{10\pi}{21} = -\sin \frac{11\pi}{21}$  and  $\sin \frac{64\pi}{21} = -\sin \frac{\pi}{21} = -\sin \frac{20\pi}{21}$ , which implies

$$\begin{aligned} \prod_{k=1}^6 \left( x - 2 \sin \frac{2^k \pi}{21} \right) \\ = x^6 - \sqrt{7} x^5 - 2x^4 + 4\sqrt{7} x^3 - 8x^2 + 1. \end{aligned}$$

## V. SPECIAL FAMILIES OF POLYNOMIALS

There exist many special families of polynomials with the well known sets of roots. However there is no any common platform connecting these families and this is, to be honest, a great challenge of our undertaking. Considering the special families of polynomials, discovered by us, we want to emphasize these ones, the origins of which can be found in Cardano's formulae (see for example [28]). So if  $\Omega_n(x)$  denote the so called  $n$ -th Chebyshev polynomial defined by the relation

$$\Omega_n(x) := 2T_n\left(\frac{x}{2}\right),$$

where  $T_n(\cos \theta) = \cos n\theta$  is the  $n$ -th Chebyshev polynomial of the first kind, then the following decompositions hold

$$(\sqrt{p})^n \Omega_n\left(\frac{x}{\sqrt{p}}\right) - q = \prod_{k=0}^{n-1} (x - a\zeta^{2k} - b\zeta^{-2k}),$$

$$(-\sqrt{-p})^n \Omega_n\left(\frac{ix}{\sqrt{p}}\right) + q = \prod_{k=0}^{n-1} (x - a\zeta^{2k+1} + b\zeta^{-2k-1}),$$

where  $\zeta = \exp(i\pi/n)$ ,  $a$  and  $b$  are any values from among the  $n$  complex values of the following roots (respectively):

$$\sqrt[n]{\frac{1}{2} \left( q + \sqrt{q^2 - 4p^n} \right)}, \quad \sqrt[n]{\frac{1}{2} \left( q - \sqrt{q^2 - 4p^n} \right)}.$$

These formulae lead to many special cases, they generate many trigonometric identities. However the disadvantage of these identities is their dependence only on two complex parameters  $p$  and  $q$ , no matter what is the value of the discussed polynomial degree. We also observe some calculating difficulties – however just on the same kind as in case of the classical Cardano's formulae.

## VI. CONCLUSIONS

Our attempt of making the algorithms, serving for symbolic determination of the complex roots of some real polynomials, more effective brought us many practical advantages. We have obtained the algorithms behaving better, at least in the cases emphasized by us, in comparison with, for example, the procedures used in Mathematica software. By the way we have generated many original identities and we have discovered new relations. We have also learned to avoid various traps appearing in the course of using the algebra of complex numbers. Considering the future plans we intend still to seek the families of polynomials with the known description of roots. Also the form of realizing our mega-algorithm, which should end our future investigations, still remains questionable.

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