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Combinatorial etude

Miroslav Stoenchev *, Venelin Todorov†‡
*Technical University of Sofia, Bulgaria
†Institute of Mathematics and Informatics, Bulgarian Academy of Sciences
‡Institute of Information and Communication Technologies, Bulgarian Academy of Sciences
Email: mrs@tu-sofia.bg, vtodorov@math.bas.bg, venelin@parallel.bas.bg,

***Abstract*—The purpose of this article is to consider a special class of combinatorial problems, the so called Prouhet-Tarry-Escot problem, solution of which is realized by constructing finite sequences of ± 1. For example, for fixed $p \in \mathbb{N}$, is well known the existence of $n_p \in \mathbb{N}$ with the property: any set of n_p consecutive integers can be divided into 2 sets, with equal sums of its p^{th}-powers. The considered property remains valid also for sets of finite arithmetic progressions of complex numbers.**

I. Morse sequence

FOR every positive integer m, let us denote with $\vartheta(m)$ and $\varrho(m)$ respectively the number of occurences of digit 1 in the binary representation of m, and the position of first digit 1 in the binary representation of m. The Morse sequence $\{a_m\}_{m=1}^{\infty}$ ([1], [4]) is defined by

$$a_m = (-1)^{\vartheta(m)+\varrho(m)-2}.$$

The following properties are derived directly:

$$a_{2^k} = (-1)^k$$

$$a_{2^k+l} = -a_l \quad \text{for} \quad l = 1, 2 \ldots, 2^k.$$

The problem of finding a number n_p, such that the set $A_{n_p} = \{1, 2, \ldots, n_p\}$ is represented as disjoint union of two subsets, say B and C, with the property:

$$\sum_{b\in B} b^p = \sum_{c\in C} c^p,$$

is solved by the sequence $\{a_m\}_{m=1}^{\infty}$. Elementary proof is given below[1] and $n_p = 2^{p+1}$ has the desired property, with

$$B = \{m \in A_{n_p} : \ a_m = 1\},$$

$$C = A_{n_p} \backslash B = \{m \in A_{n_p} : \ a_m = -1\}.$$

This result can be generalized to arbitrary arithmetic progressions of complex numbers. As example, if $a, d \in \mathbb{C}$, $d \neq 0$ and $A_{n_p} = \{a + kd : \ k = 0, 1, \ldots, n_p - 1\}$, then $n_p = 2^{p+1}$ and $B = \{a + kd \in A_{n_p} : \ a_{k+1} = 1\}$.

[1]Similar solutions and generalizations of the Prouhet-Tarry-Escot problem are considered in [2], [3], [5], [6], [8]

II. Formulation of the main results

Let us define $\{H_{n,m}(z)\}_{n,m=1}^{\infty}$, by

$$H_{n,m}(z) = \sum_{l=n}^{\infty} \sum_{k=1}^{2^l} a_k \left(P(z) + k.Q(z)\right)^m,$$

where $P, Q \in \mathbb{C}[z]$ are complex polynomials.

***Proposition** 1:* If $m = 0, 1, \ldots, n-1$ then $H_{n,m} \equiv 0$, while if n is even number the following equality is satisfied

$$H_{n,n}(z) = n! 2^{\frac{n^2-n}{2}} Q^n(z).$$

***Proposition** 2:* Let $n \in \mathbb{N}$ be a even number and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are complex numbers, then

$$\sum_{k=1}^{2^n} a_k(\alpha_1 + k)(\alpha_2 + k)\cdots(\alpha_n + k) = n! 2^{\frac{n^2-n}{2}}.$$

***Proposition** 3:* If $P \in \mathbb{C}[z]$ is a complex polynomial, then

$$\sum_{k=1}^{2^{1+\deg P}} a_k P(k) = 0.$$

***Proposition** 4:* Let p and k be positive integers. Then there exist $n \in \mathbb{N}$, $n \leq 2^{p\lceil \log_2 k \rceil}$ and distinct square-free positive integers x_{ij}, $i = 1, 2, \ldots, k$; $j = 1, 2, \ldots, n$ with the property:

$$\sum_{j=1}^{n} x_{1j}^r = \sum_{j=1}^{n} x_{2j}^r = \cdots = \sum_{j=1}^{n} x_{kj}^r, \ \forall r = 1, 2, \ldots, p.$$

Let $\{b_i\}_{i=0}^{\infty}$ and $\{d_i\}_{i=0}^{\infty}$ be arbitrary sequences of complex numbers. Let γ_n and Γ_n be the sets:

$$\gamma_m = \{b_m + kd_m : \ k \in \mathbb{Z}\} \text{ and}$$

$$\Gamma_n = \gamma_1 \gamma_2 \ldots \gamma_n = \{\prod_{i=1}^{n}(b_i + kd_i) : \ k \in \mathbb{Z} \}.$$

***Proposition** 5:* Let n and m be positve integers. Then there exists an integer $s = s(n, m)$ with the property: every $s-$element subset of Γ_n, where k runs through s consequtive integers, can be represented as disjoint union of m subsets, with equal sums of the elements in each one.

The proof of each of the formulated above propositions, with the exception for 5, is based on the following lemma:

***Lemma** 1:* Set $a, d \in \mathbb{C},\ d \neq 0,\ p \in \mathbb{N}$ and $A_{2^{p+1}} = \{a + kd:\ k = 0, 1, \ldots, 2^{p+1} - 1\}$. Then there are sets $B \cap C = \emptyset$, $B \cup C = A_{2^{p+1}}$ such that

$$\sum_{b\in B} b^p = \sum_{c\in C} c^p.$$

***Corollary** 1:* Under assumptions of lemma 1, it holds

$$\sum_{b\in B} b^r = \sum_{c\in C} c^r,\ r = 0, 1, \ldots, p.$$

To prove lemma 1 and its consequence, we define a sequence of polynomials: $\{T_{s,p}(z)\}_{s=0}^{\infty}$, through which we will gradually calculate the differences between the sums of equal powers of the elements in $B = \{a+kd \in A_{2^{p+1}}:\ a_{k+1} = 1\}$ and $C = \{a + kd \in A_{2^{p+1}}:\ a_{k+1} = -1\}$. For $s \geq 0$ set

$$T_{s,p}(z) = \sum_{k=0}^{4^{s+1}-1} a_{k+1}(z + kd)^p$$

and we calculate

$$T_{s,p}(z) =$$

$$\sum_{0\leq k\leq 4^{s+1}-1; a_{k+1}=1} (z+kd)^p - \sum_{0\leq k\leq 4^{s+1}-1;\ a_{k+1}=-1} (z+kd)^p.$$

When $s \leq \frac{p-1}{2}$, set $z = a$ to obtain

$$T_{s,p}(a) = \sum_{b\leq a+(4^{s+1}-1)d} b^p - \sum_{c\leq a+(4^{s+1}-1)d} c^p,$$

where summation is by $b \in B,\ c \in C$.
Set $p = 2m + r,\ r \in \{0, 1\}$. Here and everywhere below the summations are performed on all $b \in B$ and $c \in C$, which satisfy the corresponding inequalities.
When $r = 1$ we obtain

$$T_{m,p}(a) = \sum_{b\leq a+(4^{m+1}-1)d} b^p - \sum_{c\leq a+(4^{m+1}-1)d} c^p$$

$$= \sum_{b\leq a+(2^{p+1}-1)d} b^p - \sum_{c\leq a+(2^{p+1}-1)d} c^p$$

$$= \sum_{b\in B} b^p - \sum_{c\in C} c^p.$$

When $r = 0$:

$$T_{m-1,p}(a) = \sum_{b\leq a+(2^{2m}-1)d} b^p - \sum_{c\leq a+(2^{2m}-1)d} c^p$$

$$= \sum_{b\leq a+(2^p-1)d} b^p - \sum_{c\leq a+(2^p-1)d} c^p.$$

On the other hand

$$\sum_{a+2^pd\leq b\leq a+(2^{p+1}-1)d} b^p - \sum_{a+2^pd\leq c\leq a+(2^p-1)d} c^p$$

$$= \sum_{2^p\leq m\leq 2^{p+1}-1} a_{m+1}(a + md)^p$$

$$= \sum_{k=0}^{2^p-1} a_{2^p+k+1}\,(a + (2^p + k)d)^p$$

$$= -\sum_{k=0}^{2^p-1} a_{k+1}\,(a + (2^p + k)d)^p$$

$$= -\sum_{k=0}^{2^p-1} a_{k+1}\,((a + 2^pd) + kd)^p =$$

$$= -\sum_{k=0}^{2^{2m}-1} a_{k+1}\,((a + 2^pd) + kd)^p$$

$$= -T_{m-1,p}(a + 2^pd).$$

Therefore, for $p = 2m$ we obtain

$$\sum_{b\in B} b^p - \sum_{c\in C} c^p =$$

$$T_{m-1,p}(a) - T_{m-1,p}(a + 2^pd).$$

Summarized:

$$\sum_{b\in B} b^p - \sum_{c\in C} c^p =$$

$$\begin{cases} T_{m,p}(a), & \text{for} \quad p = 2m + 1 \\ T_{m-1,p}(a) - T_{m-1,p}(a + 2^pd), & \text{for} \quad p = 2m \end{cases}$$

III. Proof of the main results

Lemma 1 follows directly from :
***Proposition** 6:*

$$T_{m-1,p}(z) = \begin{cases} 0, & \text{when} \quad p = 2m - 1 \\ p!2^{\frac{p^2-p}{2}}d^p, & \text{when} \quad p = 2m \end{cases}$$

***Proof** 1:* Let us determine the polynomials $\{T_{s,p}(z)\}_{s=0}^{\infty}$ by finding recurrent formula. Since $a_1 = a_4 = 1,\ a_2 = a_3 = -1$, then

$$T_{0,p}(z) = (z + 3d)^p - (z + 2d)^p - (z + d)^p + z^p.$$

We will prove that for all $s \geq 1$ is valid

$$T_{s,p}(z) = T_{s-1,p}(z + 3.4^sd) - T_{s-1,p}(z + 2.4^sd)$$
$$-T_{s-1,p}(z + 4^sd) + T_{s-1,p}(z).$$

For example, if $s = 1$ then:

$$T_{1,p}(z) = \sum_{k=0}^{15} a_{k+1}(z + kd)^p$$

$$= \sum_{k=0}^{3} a_{k+1}(z + kd)^p + \sum_{k=4}^{7} a_{k+1}(z + kd)^p$$

$$+\sum_{k=8}^{11} a_{k+1}(z + kd)^p + \sum_{k=12}^{15} a_{k+1}(z + kd)^p$$

$$= T_{0,p}(z) + \sum_{m=0}^{3} a_{2^2+m+1}((z + 4d) + md)^p$$

$$+\sum_{m=0}^{3} a_{2^3+m+1}((z+2.4d)+md)^p$$

$$+\sum_{m=0}^{3} a_{2^3+2^2+m+1}((z+3.4d)+md)^p$$

$$=T_{0,p}(z)-\sum_{m=0}^{3} a_{m+1}((z+4d)+md)^p$$

$$-\sum_{m=0}^{3} a_{m+1}((z+2.4d)+md)^p$$

$$+\sum_{m=0}^{3} a_{m+1}((z+3.4d)+md)^p$$

$$=T_{0,p}(z)-T_{0,p}(z+4d)$$

$$-T_{0,p}(z+2.4d)+T_{0,p}(z+3.4d).$$

The proof is similar in the general case:

$$T_{s,p}(z)=\sum_{k=0}^{4^{s+1}-1} a_{k+1}(z+kd)^p$$

$$=\sum_{k=0}^{4^s-1} a_{k+1}(z+kd)^p+\sum_{k=4^s}^{2.4^s-1} a_{k+1}(z+kd)^p$$

$$+\sum_{k=2.4^s}^{3.4^s-1} a_{k+1}(z+kd)^p+\sum_{k=3.4^s}^{4^{s+1}-1} a_{k+1}(z+kd)^p$$

$$=T_{s-1,p}(z)+\sum_{m=0}^{4^s-1} a_{4^s+m+1}((z+4^s d)+md)^p$$

$$+\sum_{m=0}^{4^s-1} a_{2.4^s+m+1}((z+2.4^s d)+md)^p$$

$$+\sum_{m=0}^{4^s-1} a_{3.4^s+m+1}((z+3.4^s d)+md)^p$$

$$=T_{s-1,p}(z+3.4^s d)-T_{s-1,p}(z+2.4^s d)$$

$$-T_{s-1,p}(z+4^s d)+T_{s-1,p}(z),$$

whence the necessary recurrent formula is established.

In the case $1 \le s \le \left[\frac{p}{2}\right]-1$, we prove that $T_{s,p}(z)$ has the type:

$$T_{s,p}(z)=\sum_{i_1=2s}^{p-2}\ \sum_{i_2=2(s-1)}^{i_1-2}\ \sum_{i_3=2(s-2)}^{i_2-2}\cdots$$

$$\cdots\sum_{i_{s+1}=0}^{i_s-2}\binom{p}{i_1}\binom{i_1}{i_2}\binom{i_2}{i_3}\cdots\binom{i_s}{i_{s+1}}d^{p-i_{s+1}}L_{s,p}z^{i_{s+1}},$$

where

$$L_{s,p}=(3^{p-i_1}-2^{p-i_1}-1)(3^{i_1-i_2}-2^{i_1-i_2}-1)\cdots$$

$$\cdots(3^{i_s-i_{s+1}}-2^{i_s-i_{s+1}}-1)4^{i_1+i_2+\cdots+i_s-si_{s+1}}.$$

Indeed, when $s=0$ follows:

$$T_{0,p}(z)=(z+3d)^p-(z+2d)^p-(z+d)^p+z^p$$

$$=\sum_{i_1=0}^{p-2}\binom{p}{i_1}(3^{p-i_1}-2^{p-i_1}-1)d^{p-i_1}z^{i_1}.$$

Direct calculation for $T_{1,p}$ gives

$$T_{1,p}(z)=T_{0,p}(z)-T_{0,p}(z+4d)$$

$$-T_{0,p}(z+2.4d)+T_{0,p}(z+3.4d)$$

$$=\sum_{i_1=0}^{p-2}\binom{p}{i_1}(3^{p-i_1}-2^{p-i_1}-1)d^{p-i_1}$$

$$\big((z+12d)^{i_1}-(z+8d)^{i_1}-(z+4d)^{i_1}+z^{i_1}\big)$$

$$=\sum_{i_1=2}^{p-2}\binom{p}{i_1}(3^{p-i_1}-2^{p-i_1}-1)d^{p-i_1}$$

$$\big((z+12d)^{i_1}-(z+8d)^{i_1}-(z+4d)^{i_1}+z^{i_1}\big)$$

$$=\sum_{i_1=2}^{p-2}\binom{p}{i_1}(3^{p-i_1}-2^{p-i_1}-1)d^{p-i_1}$$

$$\left(z^{i_1}+\sum_{i_2=0}^{i_1}\binom{i_1}{i_2}(3^{i_1-i_2}-2^{i_1-i_2}-1)(4d)^{i_1-i_2}z^{i_2}\right)$$

$$=\sum_{i_1=2}^{p-2}\binom{p}{i_1}(3^{p-i_1}-2^{p-i_1}-1)d^{p-i_1}$$

$$\left(\sum_{i_2=0}^{i_1-2}\binom{i_1}{i_2}(3^{i_1-i_2}-2^{i_1-i_2}-1)(4d)^{i_1-i_2}z^{i_2}\right)=$$

$$=\sum_{i_1=2}^{p-2}\sum_{i_2=0}^{i_1-2}\binom{p}{i_1}\binom{i_1}{i_2}(3^{p-i_1}-2^{p-i_1}-1)$$

$$(3^{i_1-i_2}-2^{i_1-i_2}-1)4^{i_1-i_2}d^{p-i_2}z^{i_2}=$$

$$=\sum_{i_1=2}^{p-2}\sum_{i_2=0}^{i_1-2}\binom{p}{i_1}\binom{i_1}{i_2}d^{p-i_2}L_{1,p}z^{i_2},$$

hence the assertion is established for $s=1$. Suppose that for some $s \ge 2$, $T_{s-1,p}(z)$ satisfies the recurent formula and denote

$$G^{s,p}_{i_1,i_2,\ldots,i_{s+1}}=\binom{p}{i_1}\binom{i_1}{i_2}\binom{i_2}{i_3}\cdots\binom{i_s}{i_{s+1}}d^{p-i_{s+1}}L_{s,p},$$

for $s \ge 1$. Direct calculation shows:

$$T_{s,p}(z)=T_{s-1,p}(z+3.4^s d)-T_{s-1,p}(z+2.4^s d)$$

$$-T_{s-1,p}(z+4^s d)+T_{s-1,p}(z)$$

$$=\sum_{i_1=2(s-1)}^{p-2}\ \sum_{i_2=2(s-2)}^{i_1-2}\ \sum_{i_3=2(s-3)}^{i_2-2}\cdots\sum_{i_s=0}^{i_{s-1}-2}G^{s-1,p}_{i_1,i_2,\ldots,i_s}$$

$$((z+3.4^s d)^{i_s}-(z+2.4^s d)^{i_s}-(z+4^s d)^{i_s}+z^{i_s})$$
$$=\sum_{i_1=2(s-1)}^{p-2}\sum_{i_2=2(s-2)}^{i_1-2}\cdots\sum_{i_s=0}^{i_{s-1}-2}G^{s-1,p}_{i_1,i_2,\ldots,i_s}$$
$$[z^{i_s}+\sum_{i_{s+1}=0}^{i_s}\binom{i_s}{i_{s+1}}(3^{i_s-i_{s+1}}-2^{i_s-i_{s+1}}-1)$$
$$4^{s(i_s-i_{s+1})}d^{i_s-i_{s+1}}z^{i_{s+1}}]$$
$$=\sum_{i_1=2s}^{p-2}\sum_{i_2=2(s-1)}^{i_1-2}\cdots\sum_{i_s=2}^{i_{s-1}-2}G^{s-1,p}_{i_1,i_2,\ldots,i_s}\sum_{i_{s+1}=0}^{i_s-2}$$
$$\binom{i_s}{i_{s+1}}(3^{i_s-i_{s+1}}-2^{i_s-i_{s+1}}-1)4^{s(i_s-i_{s+1})}d^{i_s-i_{s+1}}z^{i_{s+1}}$$
$$=\sum_{i_1=2s}^{p-2}\sum_{i_2=2(s-1)}^{i_1-2}\cdots\sum_{i_s=2}^{i_{s-1}-2}\sum_{i_{s+1}=0}^{i_s-2}G^{s-1,p}_{i_1,i_2,\ldots,i_s}\binom{i_s}{i_{s+1}}$$
$$(3^{i_s-i_{s+1}}-2^{i_s-i_{s+1}}-1)4^{s(i_s-i_{s+1})}d^{i_s-i_{s+1}}z^{i_{s+1}}$$
$$=\sum_{i_1=2s}^{p-2}\sum_{i_2=2(s-1)}^{i_1-2}\cdots\sum_{i_s=2}^{i_{s-1}-2}\sum_{i_{s+1}=0}^{i_s-2}G^{s,p}_{i_1,i_2,\ldots,i_{s+1}}z^{i_{s+1}},$$

which prove that $T_{s,p}(z)$ satisfies the recurrent formula.

Let us determine the degree of $T_{s,p}(z),\ s\geq 0$. According to the derived formula we find $i_{s+1}\leq i_s-2\leq i_{s-1}-4\leq\cdots\leq i_1-2s\leq p-2(s+1)$, as equality is reached everywhere. Therefore $\deg T_{s,p}(z)=p-2(s+1)$. If $p=2m+r,\ r\in\{0,1\}$, then

$$\deg T_{m-1,p}(z)=p-2m=r.$$

For $r=0$ we obtain that $T_{m-1,p}(z)$ is a constant, equal to $p!2^{\frac{p^2-p}{2}}d^p$. Indeed

$$T_{m-1,p}(z)=$$
$$=\sum_{i_1=2(m-1)}^{p-2}\sum_{i_2=2(m-2)}^{i_1-2}\cdots\sum_{i_{m-1}=2}^{i_{m-2}-2}\sum_{i_m=0}^{i_{m-1}-2}G^{m-1,p}_{i_1,i_2,\ldots,i_{s+1}}z^{i_m}$$
$$=\sum_{i_1=2(m-1)}^{2(m-1)}\sum_{i_2=2(m-2)}^{2(m-2)}\cdots\sum_{i_{m-1}=2}^{2}\sum_{i_m=0}^{0}$$
$$G^{m-1,p}_{i_1,i_2,\ldots,i_{s+1}}z^{i_m}$$
$$=G^{m-1,p}_{p-2,p-4,p-6\ldots,2,0}$$
$$=\binom{p}{p-2}\binom{p-2}{p-4}\cdots\binom{4}{2}\binom{2}{0}d^pL_{m-1,p}$$
$$=\frac{p!d^p}{2^m}L_{m-1,p}=p!2^{\frac{p^2-p}{2}}d^p\Longrightarrow$$
$$\Longrightarrow T_{m-1,p}(z)=p!2^{\frac{p^2-p}{2}}d^p,\ \text{for } p=2m.$$

In the case $r=1$, we will prove that $T_{m,p}(z)=0$:

$$T_{m,p}(z)=$$
$$=T_{m-1,p}(z+3.4^m d)-T_{m-1,p}(z+2.4^m d)-$$
$$T_{m-1,p}(z+4^m d)+T_{m-1,p}(z)$$
$$=\sum_{i_1=2(m-1)}^{p-2}\sum_{i_2=2(m-2)}^{i_1-2}\cdots\sum_{i_m=0}^{i_{m-1}-2}G^{m-1,p}_{i_1,i_2,\ldots,i_m}$$
$$((z+3.4^m d)^{i_m}-(z+2.4^m d)^{i_m}-(z+4^m d)^{i_m}+z^{i_m})=0,$$

and the above equality is valid, since the summation index i_m takes values 0 and 1. Thus the proposition 6 is proved. Proofs of propositions 1,2,3,4,5 will be presented in [7].

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