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The Compositional Rule of Inference vs the Bandler-Kohout Subproduct: a Comparison of Two Standard Rules of Inference

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***Abstract*—This contribution focuses on the most popular scheme of reasoning in approximate reasoning, generalized modus ponens. Also, we consider the case when the reasoning is performed with one fuzzy rule. Usually, the compositional rule of inference introduced by Zadeh is involved. However, it is also common to use the Bandler-Kohout subproduct. We compare these two rules showing by experimental results the conditions when applying one of them is more appropriate. We concentrate on an example of image transformation where applying a different rule of inference gives a different conclusion. Moreover, we point out some theoretical justifications for particular fuzzy connectives used in both methods (fuzzy implication functions, triangular norms and, in general, fuzzy conjunctions).**

I. Introduction

WHENEVER we have imprecise data but would like to obtain meaningful results, we use methods called approximate reasoning. In this contribution, we analyse approximate reasoning based on fuzzy sets regarding one scheme, generalised modus ponens. For this scheme, we infer using the following idea,

RULE:	IF x is A, THEN y is B
FACT:	x is A'
CONCLUSION:	y is B'

where A, A', B, B' are fuzzy sets representing properties of objects x and y. A and A' are such that they are only slightly different (in some subjective opinions and using this informal language). It is why the conclusion expressed by a B' should also be "similar" to B to keep the intention of approximate reasoning. In our investigations, we consider two rules of inference:

- the Compositional Rule of Inference (CRI), see [1]

$$B'(y) := \sup_{x\in X} T(A'(x), I(A(x), B(y))), \quad y \in Y, \tag{CRI}$$

- the Bandler-Kohout Subproduct (BKS), see [2]

$$B'(y) := \inf_{x\in X} I(A'(x), T(A(x), B(y))), \quad y \in Y, \tag{BKS}$$

where T is a t-norm or a generalization of a conjunction and I is a fuzzy implication. We analyse particular sample data in order to show when (CRI) is better than (BKS) and vice versa. It should be noted that various scientists study these two rules of inference, see, e.g. [3], [4]. We focus on image processing and show that using a different inference rule gives a distinct conclusion, what is reflected in the output image. Our main hypothesis is: if A and A' are quite "similar", then B and B' will be more similar when B' is obtained from (CRI). However, if A and A' are "different", then B and B' will be more similar when B' will be calculated from (BKS).

The paper is organised as follows. Section 2 recalls some necessary definitions and facts used in the sequel. In Section 3, we present some experimental results and state our conclusions, observations, and verifications of hypothesises. Section 4 presents some theoretical results that partially justify our assumptions.

II. Preliminaries

First, let us introduce a symbol $\mathcal{F}(X)$ as a family of all fuzzy sets on X. Let us start with recalling some standard definitions and facts regarding t-norms and fuzzy implications.

Definition 2.1 (see [5], [6]): A function $T\colon [0,1]^2 \to [0,1]$ is called a triangular norm (t-norm in short), if it satisfies the following conditions for all $x, y, z \in [0,1]$

(T1) $T(x,y) = T(y,x)$,
(T2) $T(x, T(y,z)) = T(T(x,y),z)$,
(T3) $T(x,y) \leq T(x,z)$ for $y \leq z$, i.e., $T(x,\cdot)$ is non-decreasing,
(T4) $T(x,1) = x$.

Theorem 2.2 (see [6, Theorem 5.1]): For a function $T\colon [0,1]^2 \to [0,1]$ the following statements are equivalent:

(i) T is a continuous Archimedean t-norm.
(ii) T has a continuous additive generator, i.e., there exists a continuous, strictly decreasing function $f\colon [0,1] \to [0,\infty]$ with $f(1) = 0$ such that

$$T(x,y) = f^{-1}\left(\min\{f(x)+f(y), f(0)\}\right), \quad x, y \in [0,1].$$

Moreover, such a representation is unique up to a positive multiplicative constant.

We will need the following characterization of convex functions.

Theorem 2.3 (see [7, Theorems 7.3.2 and 7.3.3]): If a function $f\colon [0,1] \to \mathbb{R}$ is continuous, then the following statements are equivalent:

(i) f is convex.

(ii) For all $x, y \in [0,1]$ such that $y \leq x$ and all $\varepsilon > 0$ such that $x+\varepsilon, y+\varepsilon \in [0,1]$ it holds

$$f(y+\varepsilon) - f(y) \leq f(x+\varepsilon) - f(x). \tag{1}$$

In our investigations we also use fuzzy implication functions.

Definition 2.4 (see [5], [8]): A function $I\colon [0,1]^2 \to [0,1]$ is called a fuzzy implication, if it satisfies the following conditions:

(I1) I is non-increasing with respect to the first variable,
(I2) I is non-decreasing with respect to the second variable,
(I3) $I(0,0) = I(1,1) = 1$ and $I(1,0) = 0$.

Definition 2.5 (see [8]): We say that a fuzzy implication I satisfies

(i) the identity principle, if

$$I(x,x) = 1, \quad x \in [0,1], \tag{IP}$$

(ii) the left neutrality property, if

$$I(1,y) = y, \quad y \in [0,1], \tag{NP}$$

(iii) the ordering property, if

$$x \leq y \Longleftrightarrow I(x,y) = 1, \quad x, y \in [0,1]. \tag{OP}$$

Definition 2.6 (see [8, Definition 2.5.1]): A function $I\colon [0,1]^2 \to [0,1]$ is called an R-implication if there exists a t-norm T such that

$$I(x,y) = \sup\{t \in [0,1] \mid T(x,t) \leq y\}, \qquad x, y \in [0,1]. \tag{2}$$

If I is generated from a t-norm T, then it will be denoted by I_T.

For R-implications generated from left continuous t-norms we have the following characterization.

Theorem 2.7 (cf. [8, Proposition 2.5.2]): For a t-norm T the following statements are equivalent:

(i) T is left-continuous.
(ii) A pair (T, I_T) satisfies the following residual principle

$$T(x,z) \leq y \Longleftrightarrow I_T(x,y) \geq z, \quad x, y, z \in [0,1], \tag{RP}$$

(iii) The supremum in the formula (2) is the maximum, i.e.,

$$I_T(x,y) = \max\{t \in [0,1] \mid T(x,t) \leq y\}, \quad x, y \in [0,1]. \tag{3}$$

Theorem 2.8 (see [8, Theorem 2.5.21]): If T is a continuous Archimedean t-norm with the additive generator f as given in Theorem 2.2, then

$$I_T(x,y) = f^{-1}(\max\{f(y) - f(x), 0\}), \quad x, y \in [0,1]. \tag{4}$$

III. Experimental results

Here, as we mentioned in the Introduction, we will consider the case when our set of fuzzy rules contains only one rule. Therefore the inference process will proceed exactly according to (CRI) and (BKS). Let us take two different rules that concern the same topic. In both cases we will use (CRI) and (BKS) and we will compare our results.

In this matter, we would like to compare fuzzy sets $A \in \mathcal{F}(X)$ and $B \in \mathcal{F}(Y)$. It is important to show that dependencies between them have an influence on a choice of the rule of inference (CRI) or (BKS). Keeping in mind $X \neq Y$, we cannot calculate the standard similarity measure. However, we will use this notion to construct a function which compares A and B. Throughout literature we may find different properties of similarity measures and in a consequence different sets of axioms (see [9]–[12]). Let us mention some of them which can be considered here. Let $S\colon \mathcal{F}(X)^2 \to [0,1]$.

(P1) $S(X,\emptyset) = 0,\ S(A,A) = 1, \quad A \in \mathcal{F}(X)$,
(P2) $S(A,B) = S(B,A), \quad A, B \in \mathcal{F}(X)$,
(P3) $S(A,B) = S(A_\sigma, B_\sigma),\ A, B \in \mathcal{F}(X)$, where if $A = [a_1, \ldots, a_n],\ B = [b_1, \ldots, b_n]$ then $A_\sigma = [a_{\sigma(1)}, \ldots, a_{\sigma(n)}],\ B_\sigma = [b_{\sigma(1)}, \ldots, b_{\sigma(n)}]$ and $\sigma \in S_n$ (is a permutation of {1, ..., n}).

Let us take the following two well-known similarity measures (see [9] and [12]),

$$M_1(A,B) = \begin{cases} 1, & A = B = \emptyset, \\ \frac{\sum_{i=1}^{n} \min\{A(a_i), B(b_i)\}}{\sum_{i=1}^{n} \max\{A(a_i), B(b_i)\}}, & \text{otherwise}, \end{cases}$$

and

$$M_2(A,B) = AM_{i=1}^{n}(1 - |a_i - b_i|),\ A, B \in \mathcal{F}(X),$$

where AM is the arithmetic mean.
For comparing sets $A \in \mathcal{F}(X),\ B \in \mathcal{F}(Y)$ we will assume that $|X| = |Y| = n$ and take the following function $N_1, N^1, N_2, N^2\colon \mathcal{F}(X) \times \mathcal{F}(Y) \to [0,1]$.

$$N_1(A,B) = \begin{cases} 1, & A = B = \emptyset, \\ \min\limits_{\sigma,\tau \in S_n} \frac{\sum_{i=1}^{n} \min\{A(a_{\sigma(i)}), B(b_{\tau(i)})\}}{\sum_{i=1}^{n} \max\{A(a_{\sigma(i)}), B(b_{\tau(i)})\}}, & \text{otherwise}, \end{cases}$$

$$N^1(A,B) = \begin{cases} 1, & A = B = \emptyset, \\ \max\limits_{\sigma,\tau \in S_n} \frac{\sum_{i=1}^{n} \min\{A(a_{\sigma(i)}), B(b_{\tau(i)})\}}{\sum_{i=1}^{n} \max\{A(a_{\sigma(i)}), B(b_{\tau(i)})\}}, & \text{otherwise}, \end{cases}$$

$$N_2(A,B) = \min_{\sigma,\tau \in S_n} AM_{i=1}^{n}(1 - |a_{\sigma(i)} - b_{\tau(i)}|),$$

$$N^2(A,B) = \max_{\sigma,\tau \in S_n} AM_{i=1}^{n}(1 - |a_{\sigma(i)} - b_{\tau(i)}|),$$

where $A \in \mathcal{F}(X), B \in \mathcal{F}(Y)$. For these functions we see that for instance $G(X,\emptyset) = 0 = G(\emptyset, Y)$ and $G(A,A) = 1$, where $A = [x_1, \ldots, x_n] = [y_1, \ldots, y_n]$ and $G \in \{N_1, N^1, N_2, N^2\}$. Symmetry cannot be checked because of the domain (which is not symmetric, however if we take a function defined on a domain $\mathcal{F}(Y) \times \mathcal{F}(X)$, then of course values will be equal).

Now let us consider two examples which can show the guidelines for the choice between (CRI) and (BKS). First, let us mention that our motivation for this work is the comparison of two different images which were obtained as the conclusions from (CRI) (Fig. 2) and (BKS) (Fig. 3) - each pixel was considered as an input with one fuzzy rule used in the inference process (see [13]).

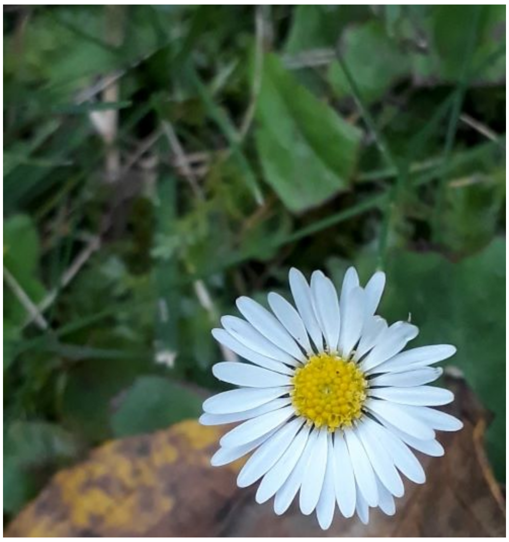

Fig. 1. The original image.

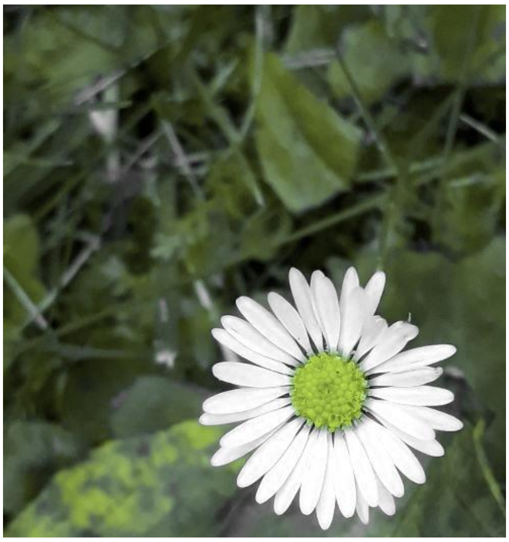

Fig. 2. Image obtained with (CRI).

In the following examples we used NumPy and Matplotlib libraries for Python (see [14], [15]). Also we have applied two pairs of (T, I):

1) $(T_{\mathbf{P}}, I_{\mathbf{GG}})$, where $T_{\mathbf{P}}(x,y) = xy$ and

$$I_{\mathbf{GG}}(x,y) = I_{T_{\mathbf{P}}}(x,y) = \begin{cases} 1, & x \le y, \\ \frac{y}{x}, & x > y, \end{cases}$$

Fig. 3. Image obtained with (BKS).

2) $(T_{\mathbf{LK}}, I_{\mathbf{LK}})$, where $T_{\mathbf{LK}}(x,y) = \max\{0, x+y-1\}$ and $I_{\mathbf{LK}}(x,y) = I_{T_{\mathbf{LK}}}(x,y) = \min\{1, 1-x+y\}$.

In both these cases, we have left-continuous t-norms and R-implications generated by corresponding t-norms.

Example 3.1: This example is directly connected with the transformations of Fig. 1 which are presented above. However, because of the quite big size of the original image, we analyse the one consisting of small parts of it (Fig. 4). It contains different colours visible in the Fig. 1 and it has 1456 pixels.

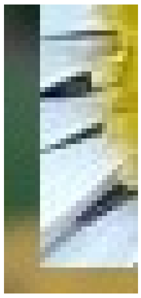

Fig. 4. Image made of pieces of the Fig. 1.

The first rule which is used by us is the following:

If an input pixel is then an output pixel is

It means: if the pixel has values $[246, 246, 81]$, then the output pixel has values $[206, 249, 88]$. Then for fuzzy sets A, B representing these values of pixels (which in general can be from the different universes) we have $N^1(A,B) = 0.915, N_1(A,B) = 0.506, N^2(A,B) = 0.935, N_2(A,B) = 0.522$, so in all cases similarity is rather high. Now let us see how the similarity of A and A' looks like compared with the one of B and B'. The results are given in the following charts. To make the plots more clear we have drawn them for every second pixel from the Fig. 4.

We can see that regardless what similarity measure is used, the similarity of $B\&B'$ is directly proportional to the similarity of $A\&A'$ for the rule (CRI) (Fig. 5, 6, 9, 10). However in the case of (BKS) the situation is not as clear as before (see Fig. 7, 8, 11, 12). Nevertheless, we might say that for many input data the similarity of $B\&B'$ is inversely proportional, in particular to data where similarity of $A\&A'$ is greater than 0.5. These conclusions can be also confirmed by the linear regression (in magenta).

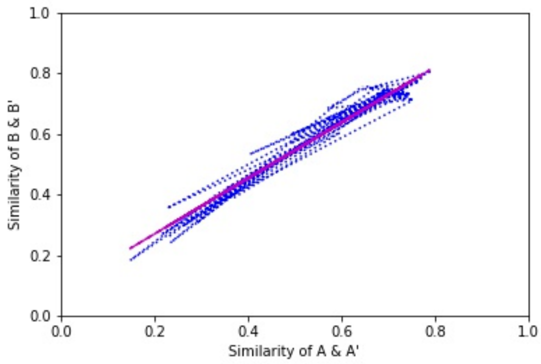

Fig. 5. Dependence between similarities calculated with M_1 for (CRI) and $(T_{\mathbf{P}}, I_{\mathbf{GG}})$, 1st rule.

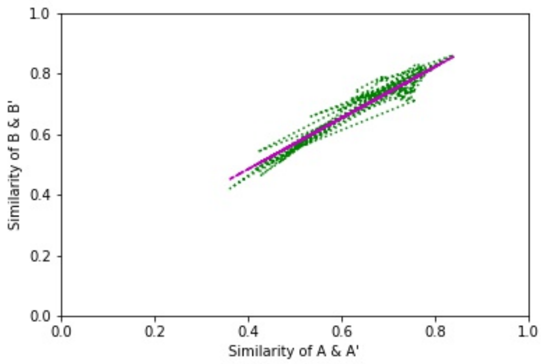

Fig. 6. Dependence between similarities calculated with M_2(CRI) and $(T_{\mathbf{P}}, I_{\mathbf{GG}})$, 1st rule.

Example 3.2: Here we consider the same Fig. 4, the same pairs $(T_{\mathbf{P}}, I_{\mathbf{GG}}), (T_{\mathbf{LK}}, I_{\mathbf{LK}})$ but we have another rule (we call it the 2nd rule):

If an input pixel is then an output pixel is 

It means: if the pixel has values $[246, 246, 81]$, then the output pixel has values $[128, 42, 239]$. Hence, for fuzzy sets A, B we have $N^1(A,B) = 0.714, N_1(A,B) = 0.346, N^2(A,B) = 0.785, N_2(A,B) = 0.372$, so in all cases similarity is lower than in Example 3.1. Now let us compare obtained similarities as we did before. On Figures 13, 14, 17, 18 we can see that values of similarities are still directly proportional. Simultaneously, we might say that for most of data obtained from BKS the similarity of $B\&B'$ is inversely proportional to the

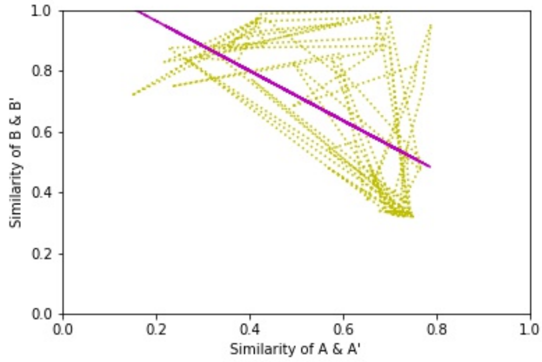

Fig. 7. Dependence between similarities calculated with M_1 (BKS) and $(T_{\mathbf{P}}, I_{\mathbf{GG}})$, 1st rule.

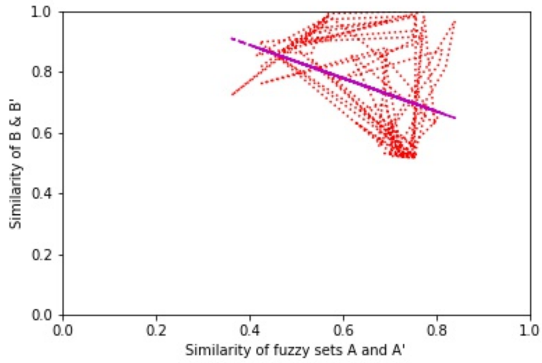

Fig. 8. Dependence between similarities calculated with M_2 (BKS) and $(T_{\mathbf{P}}, I_{\mathbf{GG}})$, 1st rule.

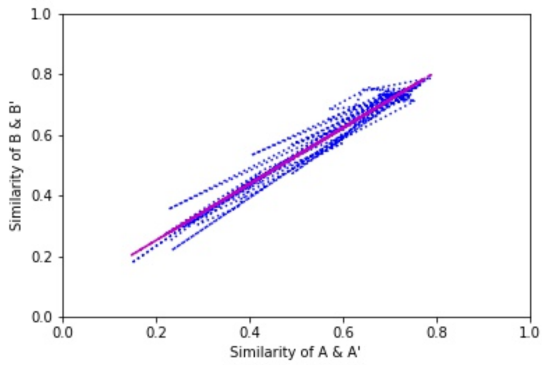

Fig. 9. Dependence between similarities calculated with M_1 for (CRI) and $(T_{\mathbf{LK}}, I_{\mathbf{LK}})$, 1st rule.

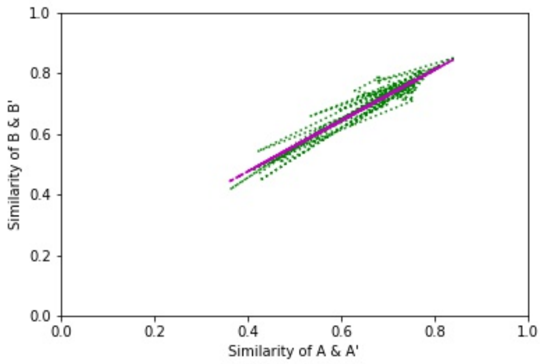

Fig. 10. Dependence between similarities calculated with M_2(CRI) and $(T_{\mathbf{LK}}, I_{\mathbf{LK}})$, 1st rule.

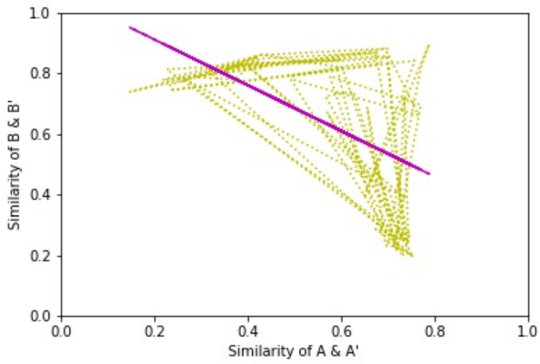

Fig. 11. Dependence between similarities calculated with M_1 (BKS) and $(T_{\mathbf{LK}}, I_{\mathbf{LK}})$, 1st rule.

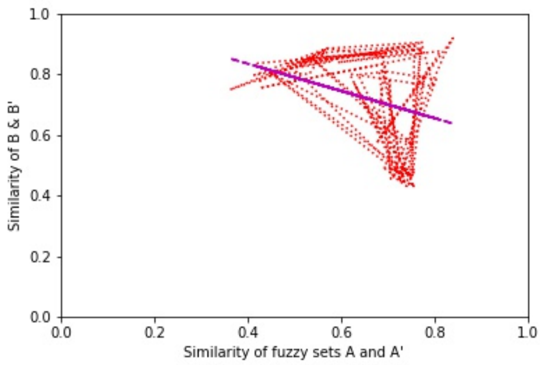

Fig. 12. Dependence between similarities calculated with M_2 (BKS) and $(T_{\mathbf{LK}}, I_{\mathbf{LK}})$, 1st rule.

similarities of $A\&A'$ (Fig. 15, 16, 19). Here the exception is only Figure 20, where we cannot say that.

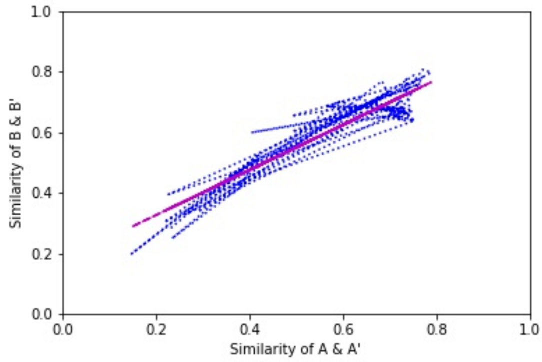

Fig. 13. Dependence between similarities calculated with M_1 for (CRI) and $(T_{\mathbf{P}}, I_{\mathbf{GG}})$, 2nd rule.

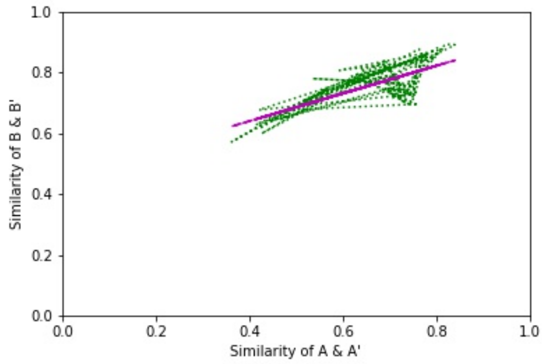

Fig. 14. Dependence between similarities calculated with M_2(CRI) and $(T_{\mathbf{P}}, I_{\mathbf{GG}})$, 2nd rule.

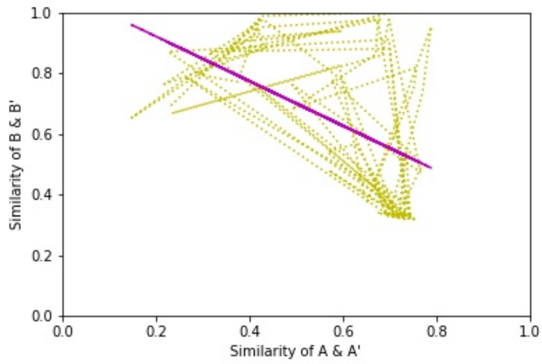

Fig. 15. Dependence between similarities calculated with M_1 (BKS) and $(T_{\mathbf{P}}, I_{\mathbf{GG}})$, 2nd rule.

After these examples we state the following observations, which are not what we expected at the beginning.

Hypothesis 1: The more A and B are similar, the more B and B' are similar for (CRI).

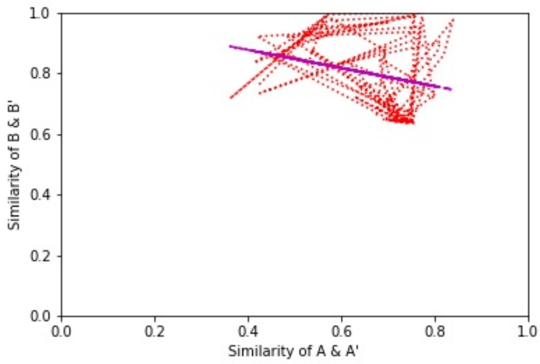

Fig. 16. Dependence between similarities calculated with M_2 (BKS) and $(T_{\mathbf{P}}, I_{\mathbf{GG}})$, 2nd rule.

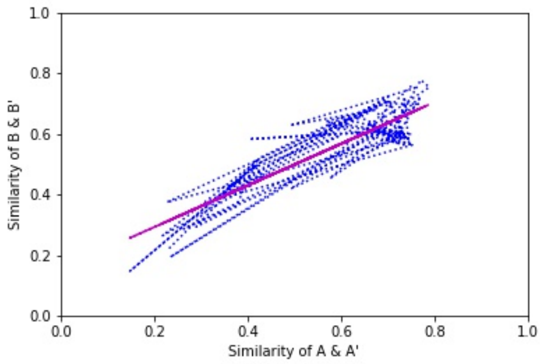

Fig. 17. Dependence between similarities calculated with M_1 for (CRI) and $(T_{\mathbf{LK}}, I_{\mathbf{LK}})$, 2nd rule.

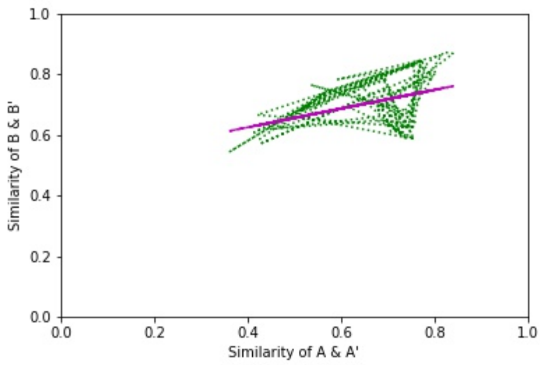

Fig. 18. Dependence between similarities calculated with M_2(CRI) and $(T_{\mathbf{LK}}, I_{\mathbf{LK}})$, 2nd rule.

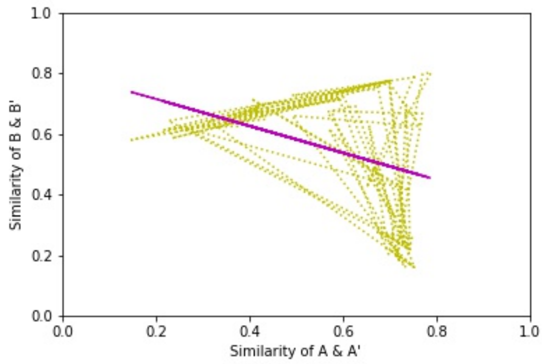

Fig. 19. Dependence between similarities calculated with M_1 (BKS) and $(T_{\mathbf{LK}}, I_{\mathbf{LK}})$, 2nd rule.

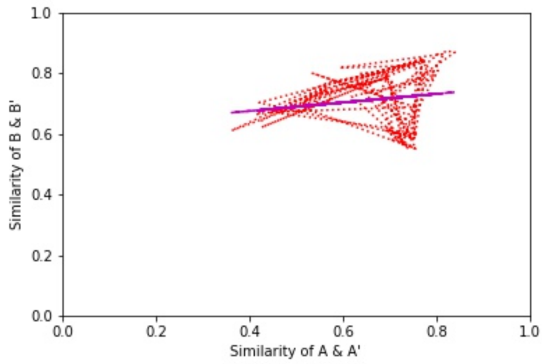

Fig. 20. Dependence between similarities calculated with M_2 (BKS) and $(T_{\mathbf{LK}}, I_{\mathbf{LK}})$, 2nd rule.

Hypothesis 2: The less A and B are similar, the more B and B' are similar for (BKS).

It turned out, it is not entirely true. Hence, our observation and conclusion are as follows.

Observation 1: Let $A, A' \in \mathcal{F}(X), B, B' \in \mathcal{F}(Y)$.

(i) The similarity of B and B' is directly proportional to the similarity of A and A' for the rule (CRI).

(ii) The similarity of B and B' is not always proportional to the similarity of A and A' for the rule (BKS).

(iii) The similarity of B and B' is usually inversely proportional to the similarity of A and A' for the rule (BKS).

IV. Theoretical Part

In this section, we want to justify the point (i) from Observation 1.

Let us consider the case of R-implications generated from left-continuous t-norms. First of all let us recall that such pairs (T, I_T), where T is a left-continuous t-norm, satisfy

$$y = \sup_{x\in[0,1]} T(x, I(x,y)), \quad y \in [0,1], \qquad \text{(CRI-GMP)}$$

which can be seen as a generalization of the property of the interpolativity (see [13]).

Let us focus on the formula (CRI). Our initial assumption is A and A' express the fact there is small difference between some property of an object x.
Let us suppose that $|X| = |Y| = n, n \in \mathbb{N}, n > 1$ and let us denote $A = [x_1, \ldots, x_n], A' = [x'_1, \ldots, x'_n], B = [y_1, \ldots, y_n]$ and let

$$\begin{aligned} \varepsilon_i &= |x_i - y_i|, \quad i = 1, \ldots, n, \\ \delta_i &= |x_i - x'_i|, \quad i = 1, \ldots, n. \end{aligned}$$

Also suppose that if A and B are 'similar', then $|x_i - y_j| \geq \varepsilon_i,\ i \neq j$.
We will show that the following inequality holds for any Archimedean continuous t-norm T with a convex generator f (t-norms used for the experiments have convex generators),

$$y_i + c \leq T(x'_i, I_T(x_i, y_i)) \leq y_i + a,$$

for $a \in \{-\varepsilon_i - \delta_i, 0, -\varepsilon_i + \delta_i\}, c \in \{-\varepsilon_i - \delta_i, -\delta_i, -\varepsilon_i + \delta_i\}$ and x_i such that $x_i - \varepsilon_i - \delta_i \geq 0,\ i = 1, \ldots, n$.
Firstly, let us consider the case $\delta_i = x_i - x'_i$.

1) if $x_i \leq y_i$, then we have

$$\begin{aligned} T(x_i - \delta_i, I_T(x_i, y_i)) &= T(x_i - \delta_i, 1) \\ &= x_i - \delta_i \\ &= y_i - \varepsilon_i - \delta_i, \end{aligned}$$

and

$$y_i - \varepsilon_i - \delta_i = x_i - \delta_i \leq T(x_i - \delta_i, I_T(x_i, y_i)).$$

2) if $x_i > y_i$, then

$$\begin{aligned} &T(x_i - \delta_i, I_T(x_i, y_i)) \leq y_i \iff \\ &\qquad I_T(x_i - \delta_i, I_T(x_i, y_i)) \geq I_T(x_i, y_i), \end{aligned}$$

which is true from the (RP) and the monotonicity of I_T. Now we will show $y_i - \delta_i \leq T(x_i - \delta_i, I_T(x_i, y_i))$. Let us recall inequality (1), which can rewritten in the following way

$$f(y + \varepsilon) + f(x) \leq f(x + \varepsilon) + f(y), \quad \text{where } y \leq x.$$

This can be applied here as

$$f(x_i - \delta_i) + f(x_i - \varepsilon_i) \leq f(x_i) + f(x_i - \varepsilon_i - \delta_i),$$

where $x := x_i - \varepsilon_i, \varepsilon = \varepsilon_i, y := x_i - \varepsilon_i - \delta_i$. The above inequality is equivalent to

$$\begin{gathered} f(x_i - \delta_i) + f(x_i - \varepsilon_i) - f(x_i) \leq f(x_i - \varepsilon_i - \delta_i) \\ \iff \\ f^{-1}(f(x_i - \delta_i) + f(x_i - \varepsilon_i) - f(x_i)) \geq x_i - \varepsilon_i - \delta_i \end{gathered}$$

Note that $x_i - \varepsilon_i - \delta_i \geq 0$, so

$$f(x_i - \delta_i) + f(x_i - \varepsilon_i) - f(x_i) \leq f(0)$$

and

$$\begin{aligned} &\min\{f(0), f(x_i - \delta_i) + f(x_i - \varepsilon_i) - f(x_i)\} \\ &\qquad = f(x_i - \delta_i) + f(x_i - \varepsilon_i) - f(x_i), \end{aligned}$$

so further we may write

$$\begin{gathered} T(x_i - \delta_i, f^{-1}(f(x_i - \varepsilon_i) - f(x_i))) \geq x_i - \varepsilon_i - \delta_i \\ \iff \\ T(x_i - \delta_i, I_T(x_i, x_i - \varepsilon_i)) \geq x_i - \varepsilon_i - \delta_i \end{gathered}$$

Now, let $x'_i > x_i$, so $\delta_i = x'_i - x_i$.

1) if $x_i \leq y_i$, then we have

$$T(x'_i, I_T(x_i, y_i)) = x'_i = y_i + \delta_i - \varepsilon_i.$$

2) if $x_i > y_i$, then we have

$$\begin{aligned} T(x'_i, I_T(x_i, y_i)) &= T(x'_i, I_T(x'_i - \delta_i, x'_i - \delta_i - \varepsilon_i)) \\ &\leq x'_i, \end{aligned}$$

which, by (RP), is equivalent to

$$1 = I_T(x'_i, x'_i) \geq I_T(x_i - \delta_i, x'_i - \delta - \varepsilon_i)$$

and

$$T(x'_i, I_T(x_i, y_i)) \leq x'_i = y_i - \varepsilon_i + \delta_i.$$

Moreover,

$$x_i + \delta_i - \varepsilon_i \leq T(x'_i, I_T(x_i, y_i)).$$

Indeed, again using the property of convex continuous function from (1) we have

$$f(x + \varepsilon) + f(y) \geq f(x) + f(y + \varepsilon),$$

and applying it for the generator f of a t-norm T we obtain

$$f(x_i + 2\delta_i) + f(x_i - \varepsilon_i - \delta_i) \geq f(x_i + \delta_i) + f(x_i - \varepsilon_i),$$

for such substitutions:
$x := x_i + \delta_i$,
$y := x_i - \varepsilon_i - \delta_i$,
$\varepsilon := \delta_i$.
Next, from the fact f is strictly decreasing we may write $f(x_i) + f(x_i - \varepsilon_i - \delta_i) \geq f(x_i + 2\delta_i) + f(x_i - \varepsilon_i - \delta_i)$.
Therefore we have

$$f(x_i - \delta_i - \varepsilon_i) + f(x_i) \geq f(x_i + \delta_i) + f(x_i - \varepsilon_i),$$

that is equivalent to

$$\begin{gathered} f(x_i - \delta_i - \varepsilon_i) \geq f(x_i + \delta_i) + f(x_i - \varepsilon_i) - f(x_i) \\ \iff \\ x_i - \delta_i - \varepsilon_i \leq f^{-1}(f(x_i + \delta_i) + f(x_i - \varepsilon_i) - f(x_i)) \\ \iff \\ x_i - \delta_i - \varepsilon_i \leq T(x_i + \delta_i, f^{-1}(f(x_i - \varepsilon_i) - f(x_i))) \\ \iff \\ x_i - \delta_i - \varepsilon_i \leq T(x_i + \delta_i, I_T(x_i, x_i - \varepsilon_i)) \\ \iff \\ y_i - \delta_i \leq T(x'_i, I_T(x_i, y_i)) \end{gathered}$$

Here again, we used the fact that

$$f(x_i + \delta_i) + f(x_i - \varepsilon_i) - f(x_i) \leq f(x_i - \delta_i - \varepsilon_i) \leq f(0).$$

The conclusion is the following: for inferred $B' = [y'_1, \ldots, y'_n]$ values of y'_i for $i \in \{1, \ldots, n\}$ are in the neighbourhood of y_i and if ε_i, δ_i approach 0, y'_i also approaches y_i and in the consequence value of the similarity measure of B and B' is close to 1.

V. Conclusions

In this contribution, we have compared two rules of inference, the Compositional Rule of Inference and the Bandler-Kohout Subproduct. Our goal was to investigate some dependencies between input and output. Our observations are the following. The similarity of B and B' is directly proportional to the similarity of A and A' for the rule (CRI). The similarity of B and B' is not always proportional to the similarity of A and A' for the rule (BKS). The similarity of B and B' is usually inversely proportional to the similarity of A and A' for the rule (BKS), especially if the similarity of A and A' (the antecedent and the input) is greater than $0,5$. In future work, we want to study these methods deeply with more rules and for different fuzzy logical operations classes.

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