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Algebraic structures gained from rough approximation in incomplete information systems

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***Abstract*—We give an algebraic approach for defining rough sets on incomplete information systems. The constructed approximation sets are based on objects. Given several attributes, the value of each attribute can be known or unknown for each object. In the current paper, we introduce four different approaches, a real value, a binary, a ternary and a likelihood approach. Furthermore, we define operations on the elements of the introduced approximation sets. For all three cases we can show that the achieved structure is a quasi-Brouwer-Zadeh distributive lattice with the defined operations. We also show that the introduced lower and upper approximations build up commutative monoids with the introduced operations.**

I. Introduction

DIFFERENT systems of rough set theory were created in the last forty years: Pawlak's original theory of rough sets (see in e.g. [1]–[3]), covering systems relying on tolerance relations [4], general covering systems [5], [6], decision-theoretic rough set theory [7], general partial approximation spaces [8], similarity based approximation spaces [9]. There is a very important common property:

- all systems rely on given background knowledge represented by the system of base sets;
- one cannot say more about an arbitrary set (representing a 'new' property) or about its members than the lower and upper approximations of the set make possible.

The members of a base set have to be treated in the same way

- absolutely in the Pawlak's original theory,
- relatively in the systems with non pairwise disjoint base sets.

It means generally that if something holds for a given object, then it holds for all objects belonging to at least one same base set containing the given object. Several researchers have been considering algebraic structures of rough sets, e.g. [10], [11], [12].

In real practice there is a huge amount of objects, and the background knowledge corresponds to an information system. The framework of an information system is given by attributes, and their possible attribute values. An object can be embedded in an information system by giving the attribute values of the attributes of the information system. In many practical cases some attribute values of an object are unknown and so these values are missing, therefore the information system is not complete. Recently researchers constructed partial approximation spaces for rough sets based on an incomplete information system [13]. In the current paper we aim to give an approach for rough sets based on

- an incomplete information system, especially on objects whose properties of certain attributes can be known or not known;
- taking into consideration all possible systems of attribute values (not only those for which there is an object in the information system with a system of attribute values).

We characterize a set by the approximation of its members but we focus on their systems of attribute values, which is a new idea in Rough Set Theory. The lower and upper approximations are given with the help of possible systems of attribute values (and not by the objects appearing in the information system).

The paper is organized as follows. In Section II we introduce all necessary definitions. In Section III we introduce operations and in Section IV we show that the defined sets build a quasi-BZ distributive lattice. In Section V we show some arithmetics of the introduced lower and upper approximation sets according to the introduced operations. Finally, in Section VI we draw a first conclusion of the new definitions and proven propositions and give some ideas for further research.

II. Approximation by rough sets

Given a set of objects Ω, we assume that each object $\omega \in \Omega$ can be characterized by n attributes. We map a set of values Σ to each attribute. For our purposes, we assume that the elements of Σ are numbers, but the reader should be aware of the fact that Σ can be any nonempty set. Each attribute assigns a value of Σ for each object. Thus each object can be represented by a vector of length n, each coordinate representing the value of one attribute.

Definition 2.1: Let U be non-empty universe of objects, $(P_1, \ldots, P_n)$ be a system of attributes $(n \in \mathbb{N}, n > 0)$, Σ be a set of values, ϵ be a fixed distinguished member of Σ and $f : U \Rightarrow \Sigma^n$ be a function which maps each object to a vector of length n. Then $I = (U, P, \Sigma, f)$ is an *information system.* If for each $i (1 \leq i \leq n)$ and $o \in U$ we have $f(o)_i \neq \epsilon$, then we say that I is *complete*, else I is said to be an *incomplete information system*.

At this point, we need to remark that this representation is not injective, which means that there can be two different objects with the same vector of attributes. (The reader can think for example of two people with the same hair colour and height.) Our purpose is to give several different approaches for a rough set approximation, depending on Σ. It depends strongly on the given problem which of the following approaches is the most suitable. Therefore we give a small motivating example for each approach.

A. The real value approximation

Let Ω be a set of objects, $\omega \in \Omega$ and $\Sigma = [0,1]$ the closed interval between 0 and 1. Further, let ν_ω be the vector belonging to the object ω,

$$\nu_\omega = (\nu_1, \ldots, \nu_n)$$

where ν_i denotes the value of the i^{th} attribute of the object ω and $\nu_i \in \Sigma$, $\forall i \in \{1, \ldots, n\}$. We determine

$$\nu_i = 0 \Leftrightarrow \text{ we know nothing about the } i^{th} \text{ attribute of } \omega,$$

i.e. 0 is the fixed distinguished member of Σ. In the following, for the convenience of the reader we write ν instead of ν_ω.

Now we can define two sets for each object ω : In one set we collect all those n-long vectors which have the same "known information" and in the other set we collect all those vectors which have the same "unknown information".

Definition 2.2: Let $\nu = (\nu_1, \ldots, \nu_n)$ be a string of length n, such that $\nu_i \in \Sigma$ and $\nu_i = 0 \Leftrightarrow$ the information about the i^{th} attribute is not known - for all $i \in \{1, \ldots, n\}$. Then we define

$$\varphi_\omega = \{v = (v_1, \ldots, v_n) \in \Sigma^n |$$
$$v_j = \nu_j \Leftrightarrow \nu_j \neq 0 \text{ and } 0 < v_i \leq 1 \Leftrightarrow \nu_i = 0\}$$

and

$$\delta_\omega = \{v = (v_1, \ldots, v_n) \in \Sigma^n |$$
$$0 < v_i \leq \nu_i \Leftrightarrow \nu_i \neq 0 \text{ and } v_i = 0 \Leftrightarrow \nu_i = 0\}.$$

Remark 1: With this definition, the sets φ_ω and δ_ω are infinite. For a set of objects Ω, we can then define the following sets ϕ and Δ, build up on φ and δ respectively, which means we collect "all possible known information" and search for the "unknown" in each object.

Definition 2.3: Let U be the universe of objects and $\Omega \subseteq U$. Then each element of Ω can be represented by an n-long vector in Σ^n. We define

$$\phi(\Omega) = \bigcup_{\omega \in \Omega} \varphi_\omega$$

and

$$\Delta(\Omega) = \bigcap_{\omega \in \Omega} \delta_\omega.$$

Remark 2: We have $(0, 0, \ldots, 0) \in \Delta(\Omega)$.

Definition 2.4: Let Ω be a set of objects. Then we say that $\mathcal{U}$ is an upper approximation of Ω if $\mathcal{U} \subseteq \phi(\Omega)$ and $\mathcal{L}$ is a lower approximation of Ω if $\mathcal{L} \subseteq \Delta(\Omega)$.

Remark 3: It is clear that there are several objects, or subsets of objects which can have the same upper and/or lower approximation.

Remark 4: The pair $(\mathcal{L},\mathcal{U})$ can be considered as a rough set.

Example 1:

Let $\nu = (0, 0.1, 0.2, 0.3, 0)$. Each ν_i denotes the value of a medical indicator, such as bloodsugar, hemoglobin, etc. Then we have

$$\varphi_\nu = \{v = (v_1, \ldots, v_5) \mid v_1, v_5 \in \Sigma,$$
$$v_2 = \nu_2, v_3 = \nu_3, v_4 = \nu_4\}$$

and

$$\delta_\nu = \{v = (v_1, \ldots, v_5) \mid v_1 = 0, v_5 = 0, v_2 = 0$$
$$\text{or } \nu_2, v_3 = 0 \text{ or } \nu_3, v_4 = 0 \text{ or } \nu_4\}.$$

B. The binary approach

Let I be a complete information system. Further, let $\Sigma = \{0,1\}$ and ν be the vector belonging to one object,

$$\nu = (\nu_1, \ldots, \nu_n)$$

where

$$\nu_i = 0 \Leftrightarrow \text{the object does not fulfill the } i^{th} \text{ attribute}$$

and

$$\nu_i = 1 \Leftrightarrow \text{ the object fulfills the } i^{th} \text{ attribute .}$$

Definition 2.5: Let $\nu = (\nu_1, \ldots, \nu_n)$ be a string of length n, such that $\nu_i \in \{0,1\}$ and $\nu_i = 1 \Leftrightarrow$ the information about the i^{th} attribute is fulfilled. Then we define φ_ν and δ_ν in the following way

$$\varphi_\nu = \{v = (v_1, \ldots, v_n) | v_j = 1 \Leftrightarrow \nu_j = 1 \text{ and}$$
$$v_i \in \{0,1\} \Leftrightarrow \nu_i = 0\}$$

and

$$\delta_\nu = \{v = (v_1, \ldots, v_n) | v_j \in \{0,1\} \Leftrightarrow \nu_j = 1 \text{ and}$$
$$v_i = 0 \Leftrightarrow \nu_i = 0\}.$$

Definition 2.6: Let Ω be a set of objects, each element is represented by an n-long vector in Σ^n. Then we define

$$\phi(\Omega) = \{(v_1, \ldots, v_n) \mid \text{if } \exists \omega \in \Omega : \nu_\omega[i] = 0 \Rightarrow v_i = 0 \text{ or } 1,$$
$$\text{otherwise } v_i = 1\}$$

and

$$\Delta(\Omega) = \{(v_1, \ldots, v_n) \mid \text{if } \exists \omega \in \Omega : \nu_\omega[i] = 1 \Rightarrow v_i = 0 \text{ or } 1,$$
$$\text{otherwise } v_i = 0\}.$$

Remark 5: In this case $\phi(\Omega)$ and $\Delta(\Omega)$ are finite sets, since $| \phi(\Omega) | \leq 2^n$ and $| \Delta(\Omega) | \leq 2^n$.

Of course, in this case we can also use the definitions of upper and lower approximations as in Definition 2.4, thus we get a similar rough set as in Remark 4.

Example 2: Let $\nu = (\nu_1, \nu_2)$, where ν_1 denotes if Disease X test is positive and ν_2 denotes if Disease X antigen is positive. Thus $\nu = (0,1)$ means that current Disease X infection is not fulfilled, and antigen exists. Then $\varphi = \{(0,1),(1,1)\}$ and $\delta = \{(0,1),(0,0)\}$.

C. The ternary approach

If we want to use the simplicity of a binary approach for an incomplete information system, then we collide with the problem that "no" and "not known" would be the same. Thus it is advisable to include a third symbol for "not known" and therefore we come to a ternary approach.
Let I be an incomplete information system, Further, let $\Sigma = \{0, \frac{1}{2}, 1\}$ and ν be the vector belonging to one object,

$$\nu = (\nu_1, \ldots, \nu_n)$$

where

$$\nu_i = 0 \Leftrightarrow \text{the object does not fulfill the } i^{th} \text{ attribute of } \nu$$

and

$$\nu_i = 1 \Leftrightarrow \text{ the object fulfills the } i^{th} \text{ attribute}$$

and

$$\nu_i = \frac{1}{2} \Leftrightarrow \text{ we don't know if the } i^{th} \text{ attribute is fulfilled.}$$

Definition 2.7: Let $\nu = (\nu_1, \ldots, \nu_n)$ be a string of length n, such that $\nu_i \in \{0, \frac{1}{2}, 1\}$ and $\nu_i = 1 \Leftrightarrow$ the information about the i^{th} attribute is fulfilled. Then we define φ_ν and δ_ν in the following way

$$\varphi_\nu = \{v = (v_1, \ldots, v_n) | v_j \geq \nu_j\}$$

and

$$\delta_\nu = \{v = (v_1, \ldots, v_n) | v_j \leq \nu_j\}.$$

Example 3: Let $\nu = (\nu_1, \nu_2, \nu_3)$, where ν_1 denotes if Disease X test is positive, ν_2 denotes if Disease Y test is positive and ν_3 denotes if Disease Z test is positive. Let $\nu = (1, 0, \frac{1}{2})$. Then

$$\varphi = \left\{ \left(1, 0, \frac{1}{2}\right), \left(1, \frac{1}{2}, \frac{1}{2}\right), \left(1, 1, \frac{1}{2}\right), (1, 0, 1), \left(1, \frac{1}{2}, 1\right), \left(1, 1, \frac{1}{2}\right), (1, 1, 1) \right\}$$

and

$$\delta = \left\{ \left(1, 0, \frac{1}{2}\right), \left(\frac{1}{2}, 0, \frac{1}{2}\right), \left(0, 0, \frac{1}{2}\right), (0, 0, 0), (1, 0, 0) \right\}.$$

Remark 6: The ternary approach enables us to investigate objects or a set of objects through attributes which can be known or unknown. For example, we can draw consequences of the relation between the infections with two or more diseases, although we do not have the exact information about each infection for every member of the set.

D. The likelihood approach

Let $\Sigma = [0, 1]$ and ν be the vector belonging to one object, $\nu = (\nu_1, \ldots, \nu_n)$ where ν_i denotes the likelihood of ν fulfilling the i^{th} attribute. In this case $\nu \in \Sigma^n$.
Since the likelihood is a real value between 0 and 1, this case is mathematically the same as in Section II-A. The likelihood approach is for example useful if the value of an attribute is binary and we know that an object has value 1 for an attribute with the likelihood p, where $p \in \Sigma$.
Example 4: Let $\nu = (\nu_1, \nu_2)$, where ν_1 denotes if Disease X antigen exists, ν_2 denotes if Disease X test is positive. Then $a = (0.4, 0.2)$ means that the object a has Covid antigen with 40% probability and has active Covid infection with 20% probability. If $b = (0.2, 0.2)$, then $\mathcal{L} = \{(x, y) \mid 0.1 \leq x \leq 0.2 \text{ and } 0.01 \leq y \leq 0.2\}$ is a lower approximation set for both, a and b.
All of our approaches have in common that the sets $\phi(\Omega)$ and $\Delta(\Omega)$ contain information about Ω without containing Ω itself. Therefore we are able to say something about a set without knowing its elements, which is a good achievement with many applications in real-life-problems.

III. OPERATIONS

In this section we introduce four operations on vectors. These operations can be applied to any two vectors of the same length, which implies they can be applied to our representation of objects for any attribute set Σ.
Definition 3.1: Let a and b be two objects, each can be represented by a vector of length $n : a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$, where $a_i, b_i \in \Sigma$ for some nonempty set Σ. Then we define the following operations:

$$a \vee b := (\max\{a_1, b_1\}, \ldots, \max\{a_n, b_n\})$$

and

$$a \wedge b := (\min\{a_1, b_1\}, \ldots, \min\{a_n, b_n\}).$$

Definition 3.2: For any $a = (a_1, \ldots, a_n) \in \Sigma^n$ we define the following Kleene complementation:

$$' : \Sigma^n \to \Sigma^n, \; a' = (1 - a_1, \ldots, 1 - a_n)$$

and the following Brouwer complementation:

$$^\sim : \Sigma^n \to \Sigma^n, \; a_i^\sim = \begin{cases} 0, & \text{if } a_i \neq 0 \\ 1, & \text{if } a_i = 0 \end{cases} .$$

IV. QUASI-BROUWER-ZADEH DISTRIBUTIVE LATTICE

In [10] quasi-Brouwer-Zadeh (BZ) distributive lattices were introduced. In this section we show that the set of objects which we introduced in Section II is a quasi-BZ lattice under the operations defined in Section III.
Proposition 4.1: $\langle \Sigma^n, \vee, \wedge, ', ^\sim, 0, 1 \rangle$ is a quasi-Brouwer-Zadeh distributive lattice in both cases $\Sigma = [0, 1]$ and $\Sigma = \{0, 1\}$.
Proof. We show that all properties of a quasi-BZ distributive lattice listed in ([10], Section 3) are fulfilled. For the convenience of the reader, we use the same notations for the

fulfilled points as the authors in Definition 7 in [10]. All steps can be achieved by direct computation. Σ is a distributive lattice with respect to $\vee$ and $\wedge$, since $\min\{a_i, \max\{b_i, c_i\}\} = \max\{\min\{a_i, b_i\}, \min\{a_i, c_i\}\}$ (see Table I).

TABLE I
VERIFICATION TABLE 1

case	$\min\{a_i, \max\{b_i, c_i\}\}$	$\max\{\min\{a_i, b_i\}, \min\{a_i, c_i\}\}$
$a_i \le b_i \le c_i$	a_i	a_i
$a_i \le c_i \le b_i$	a_i	a_i
$b_i \le a_i \le c_i$	a_i	a_i
$b_i \le c_i \le a_i$	c_i	c_i
$c_i \le a_i \le b_i$	a_i	a_i
$c_i \le b_i \le a_i$	b_i	b_i

The property (K1) is fulfilled, since $1-(1-a_i) = a_i$. We verify (K2) by II. (K2) $(a \vee b)' = a' \wedge b'$ means in our case $1-\max\{a_i, b_i\} = \min\{(1-a_i), (1-b_i)\}$ must be fulfilled in all possible cases (see Table II).

TABLE II
VERIFICATION TABLE 2

	$1-\max\{a_i, b_i\}$	$\min\{(1-a_i), (1-b_i)\}$
$a_i \le b_i \le c_i$	$1-b_i$	$1-b_i$
$a_i \le c_i \le b_i$	$1-b_i$	$1-b_i$
$b_i \le a_i \le c_i$	$1-a_i$	$1-a_i$
$b_i \le c_i \le a_i$	$1-a_i$	$1-a_i$
$c_i \le a_i \le b_i$	$1-b_i$	$1-b_i$
$c_i \le b_i \le a_i$	$1-a_i$	$1-a_i$

Further, property (K3) $a \wedge a' \le b \vee b'$ is fulfilled if and only if $\min\{a_i, 1-a_i\} \le \max\{b_i, 1-b_i\}$ in all possible cases. This is clear since $\min\{a_i, 1-a_i\} \le 0.5 \le \max\{b_i, 1-b_i\}$. Furthermore, the property (B1) $a \wedge a^{\sim\sim} = a$ is fulfilled since $\min\{a_i, 1\} = a_i$ and the property (B2) $(a \vee b)^{\sim} = a^{\sim} \wedge b^{\sim}$ is fulfilled since $(a \vee b)^{\sim}[i] = 1 \Leftrightarrow a_i = b_i = 0$ and $(a^{\sim} \wedge b^{\sim})[i] = 1 \Leftrightarrow a_i = b_i = 0$. Finally, (B3) $a \wedge a^{\sim} = 0$ holds since either $a_i = 0$ or $a_i^{\sim} = 0$. □

We can even say more about this quasi-BZ lattice, which the next proposition will show.

Proposition 4.2: $\langle \Sigma^n, \vee, \wedge, ', ^{\sim}, 0, 1\rangle$ is a de Morgan BZ (BZ^{dM}) distributive lattice.

Proof. We have

$$(a \wedge b)_i^{\sim} = \begin{cases} 0 \Leftrightarrow a_i \wedge b_i \neq 0 \Leftrightarrow \min(a_i, b_i) \neq 0 \\ 1 \Leftrightarrow a_i \wedge b_i = 0 \Leftrightarrow \min(a_i, b_i) = 0 \end{cases}$$

and

$$(a^{\sim} \vee b^{\sim})_i = \begin{cases} 0 \Leftrightarrow \max(a_i^{\sim}, b_i^{\sim}) = 0 \Leftrightarrow \\ a_i^{\sim} \wedge b_i^{\sim} = 0 \Leftrightarrow a_i \neq 0, b_i \neq 0 \\ \hline 1 \Leftrightarrow \max(a_i^{\sim}, b_i^{\sim}) \neq 0 \Leftrightarrow \\ a_i^{\sim} \wedge b_i^{\sim} \neq 0 \Leftrightarrow a_i = 0, b_i = 0 \end{cases}.$$

Thus $\langle \Sigma^n, \vee, \wedge, ', ^{\sim}, 0, 1\rangle$ fulfills the $\vee$ de Morgan property (B2a). □

V. ARITHMETICS OF APPROXIMATION SETS

In this section we investigate the connection between the operations related to objects and the approximations introduced in Section II.

Proposition 5.1: Let Ω be a set of objects, $\omega_1, \omega_2 \in \Omega$. Then

1) $\delta_{\omega_1} \cap \delta_{\omega_2} = \delta_{\nu_{\omega_1} \wedge \nu_{\omega_2}}$
2) $\delta_{\omega_1} \cup \delta_{\omega_2} = \delta_{\nu_{\omega_1} \vee \nu_{\omega_2}}$

Proof. We denote $v = (v_1, \ldots, v_n)$ and v_i denotes the i^{th} coordinate of v for each $i = 1, \ldots, n$. For the convenience of the reader, we denote the i^{th} coordinate of ν_{ω_1} by $\nu_{\omega_1}[i]$ and the i^{th} coordinate of ν_{ω_2} by $\nu_{\omega_2}[i]$.

1) By definition we have $\delta_{\omega_1} \cap \delta_{\omega_2} = \{v \mid 0 \le v_i \le \nu_{\omega_1}[i] \Leftrightarrow \nu_{\omega_1}[i] \neq 0; v_i = 0 \Leftrightarrow \nu_{\omega_1}[i] = 0\} \cap \{v \mid 0 \le v_i \le \nu_{\omega_2}[i] \Leftrightarrow \nu_{\omega_2}[i] \neq 0; v_i = 0 \Leftrightarrow \nu_{\omega_2}[i] = 0\} = \{v \mid 0 \le v_i \le \min\{\nu_{\omega_1}[i], \nu_{\omega_2}[i]\} \Leftrightarrow \nu_{\omega_1}[i] \neq 0$ and $\nu_{\omega_2}[i] \neq 0; v_i = 0 \Leftrightarrow \nu_{\omega_1}[i] = 0$ or $\nu_{\omega_2}[i] = 0\}$ and since the latter case is included in the first case this is equal to $\{v \mid 0 \le v_i \le \min\{\nu_{\omega_1}[i], \nu_{\omega_2}[i]\}\}$, which is $\delta_{\nu_{\omega_1} \wedge \nu_{\omega_2}}$ by definition.
2) Similarly to the previous case we have $\delta_{\omega_1} \cup \delta_{\omega_2} = \{v \mid 0 \le v_i \le \max\{\nu_{\omega_1}[i], \nu_{\omega_2}[i]\}\} = \delta_{\nu_{\omega_1} \vee \nu_{\omega_2}}$.

□

Proposition 5.2: Let Ω be a set of objects, $\omega_1, \omega_2 \in \Omega$. Then

1) $\varphi_{\omega_1} \cap \varphi_{\omega_2} \neq \emptyset \Leftrightarrow \varphi_{\omega_1} \subseteq \varphi_{\omega_2}$ or $\varphi_{\omega_2} \subseteq \varphi_{\omega_1}$
2) $\varphi_{\nu_{\omega_1} \wedge \nu_{\omega_2}} \subseteq \varphi_{\omega_1} \cup \varphi_{\omega_2}$

Proof. We denote $v = (v_1, \ldots, v_n)$ and v_i denotes the i^{th} coordinate of v for each $i = 1, \ldots, n$. For the convenience of the reader, we denote the i^{th} coordinate of ν_{ω_1} by $\nu_{\omega_1}[i]$ and the i^{th} coordinate of ν_{ω_2} by $\nu_{\omega_2}[i]$.

1) We have $\varphi_{\omega_1} \cap \varphi_{\omega_2} = \{v \mid 0 \le v_i \le 1 \Leftrightarrow \nu_{\omega_1}[i] = 0$ and $\nu_{\omega_2}[i] = 0; v_i = \nu_{\omega_1}[i] = \nu_{\omega_2}[i]($ if $\nu_{\omega_1}[i] = \nu_{\omega_2}[i])\}$.
2) The statement is true since $\varphi_{\nu_{\omega_1} \wedge \nu_{\omega_2}} = \{v \mid 0 \le v_i \le 1 \Leftrightarrow \min\{\nu_{\omega_1}[i], \nu_{\omega_2}[i]\} = 0; v_i = \min\{\nu_{\omega_1}[i], \nu_{\omega_2}[i]\}\}$.

□

Let $\mathcal{L}$ and $\mathcal{U}$ be a lower and upper approximation of a set Ω as defined in Definition 2.4. We denote the algebraic closure of $\mathcal{L}$ by $\overline{\mathcal{L}}$ and the algebraic closure of $\mathcal{U}$ by $\overline{\mathcal{U}}$. Then we gain a commutative monoid for these algebraic closures with the operations defined in Definition 3.1. Thus we can prove the following propositions.

Proposition 5.3: Let Ω be a set of objects and $\omega \in \Omega$. If $\underbrace{(1, \ldots, 1)}_{n} \in \overline{\mathcal{L}}$, then $\langle \overline{\mathcal{L}}, \wedge\rangle$ is a commutative monoid.

Proof. The operation $\wedge$ is associative since $\min\{a, \min\{b, c\}\} = \min\{\min\{a, b\}c\}$. Further we have a unit element $e = \underbrace{(1, \ldots, 1)}_{n}$ since $\min\{a, 1\} = a$ for all $a \in [0, 1]$. By assumption we have $e \in \overline{\mathcal{L}}$. Finally, the operation is commutative since $\min\{a, b\} = \min\{b, a\}$.

□

Proposition 5.4: Let Ω be a set of objects and $\omega \in \Omega$. If $\underbrace{(0, \ldots, 0)}_{n} \in \overline{\mathcal{L}}$, then $\langle \overline{\mathcal{L}}, \vee\rangle$ is a commutative monoid.

Proof. The operation $\vee$ is associative since $\max\{a, \max\{b, c\}\} = \max\{\max\{a, b\}c\}$. Further we have a unit element $e = \underbrace{(0, \ldots, 0)}_{n}$ since $\max\{a, 0\} = a$ for all $a \in [0, 1]$. By assumption we have $e \in \overline{\mathcal{L}}$. Finally, the operation is commutative since $\max\{a, b\} = \max\{b, a\}$. □

Proposition 5.5: Let Ω be a set of objects and $\omega \in \Omega$. If $\underbrace{(1, \ldots, 1)}_{n} \in \overline{\mathcal{U}}$, then $\langle \overline{\mathcal{U}}, \wedge \rangle$ is a commutative monoid.

Proof. The operation $\wedge$ is associative since $\min\{a, \min\{b, c\}\} = \min\{\min\{a, b\}c\}$. Further we have a unit element $e = \underbrace{(1, \ldots, 1)}_{n}$ since $\min\{a, 1\} = a$ for all $a \in [0, 1]$. By assumption we have $e \in \overline{\mathcal{U}}$. Finally, the operation is commutative since $\min\{a, b\} = \min\{b, a\}$. □

Proposition 5.6: Let Ω be a set of objects and $\omega \in \Omega$. If $\underbrace{(0, \ldots, 0)}_{n} \in \overline{\mathcal{U}}$, then $\langle \overline{\mathcal{U}}, \vee \rangle$ is a commutative monoid.

Proof. The operation $\vee$ is associative since $\max\{a, \max\{b, c\}\} = \max\{\max\{a, b\}c\}$. Further we have a unit element $e = \underbrace{(0, \ldots, 0)}_{n}$ since $\max\{a, 0\} = a$ for all $a \in [0, 1]$. By assumption we have $e \in \overline{\mathcal{U}}$. Finally, the operation is commutative since $\max\{a, b\} = \max\{b, a\}$. □

VI. Conclusion

In the current paper, we introduce a representation for objects of an incomplete information system and we introduce operations which make the gained structure a BZ-distributive lattice. Since in ([10] Proposition 14) a complete information system is associated to a BZ-distributive lattice, this opens the door to several possibilities for investigating rough sets based on incomplete information systems. From a certain point of view, an incomplete information system can be handled mathematically similarly as a complete information system. The literature of rough set theory is extremely large for complete information systems, e.g. [10], [12], [14]. Given a quasi-BZ lattice, a whole rough approximation space can be constructed (see for example Proposition 8 and Definition 8 in [10]). Therefore the current paper can be considered as a new idea how to represent an incomplete information system in such a way that known algebraic structures will be achieved. In this way, we can use good properties of algebraic structures in order to handle sets of objects, even if we do not have all important information. It seems to be surprising that we find the algebraic structure of monoids when we investigate rough sets, but actually monoids also appear in algebraic approaches for complete information systems, see for example [15] and [16]. Therefore it seems to be worthwhile to investigate monoids when we investigate rough sets. A research problem for the future is to find applications which can help to solve problems representing and working with information systems based on the algebraic structures gained from the approaches which were represented in the current paper.

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