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On some concept lattice of social choice functions

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***Abstract*—Social choice function or voting procedure is one of the crucial concepts in the domain of political sciences. It maps individuals' preferences over a set of candidates to some subset (possibly one-element) of the candidates who can be thought as the winners of an election procedure. The paper is aimed at applications of formal concept analysis methods to study of social choice functions. We will construct concept lattices over selected set of social choice functions characterized by possessing some properties deemed as important from the point of view of political sciences. We will discuss issues connected with reducibility of both objects and attributes, irreducibility of object concepts as well as attribute concepts and attribute implications. We will discuss also the shape of the constructed concept lattice of social choice functions which in some part is exceptionally regular from the perspective of the lattice theory.**

I. Introduction

THIS paper is aimed at some applications of formal concept analysis (FCA) methods [32], [6] in social choice theory [16], [4], [19], one of the most important research domains of political sciences. We concentrate on social choice functions or voting procedures which are concepts of a crucial importance in the theory of social choice [4], [5], [18]. Our aim is to offer a non-standard approach to studying and comparing popular social choice functions. The FCA has been used in broadly meant social choice theory (cf., e.g., [27], [27], [8] or even [6]) but not with such a specific goal as here. Also, various non-classical approaches has been proposed in this area. For example, fuzzy logic has been applied with success to model various aspects of social choice (cf., e.g.[9], [21], [10], [22], [12], [23], [11]) Another line of non-classical research in this area, which is relevant for our purposes, is based on the rough-sets theory (RST) [25], [24], [26]. In particular, the research presented in [3], [13] concentrated on the issue of reduction of voting criteria and on measuring of similarity and dissimilarity of different social choice functions, ideas and methods used in comparison of voting procedures.

Actually, rough set theory can be viewed as a similar theory to formal concept analysis in the domain of data mining and knowledge discovery [31], [30] and such view drove our attention to the idea of application of formal concept analysis methods in the area of social choice functions.

The rest of the paper is organized as follows. In Section 2 some basic concepts of theory of social choice functions are introduced and discussed including voting procedures and criteria used for comparisons of different voting procedures. In Section 3 formal context of social choice functions and its concept lattice are introduced together with an investigation of the structure of concept lattice of these social choice functions. Section 4 is devoted to analysis of information provided by concept lattice of social choice functions, including attribute independency, reduction of information and attribute implications holding in the analyzed context of voting procedures. Section 4 is followed by Conclusions discussing results and presenting directions for further research.

II. Social choice functions

We consider social choice problem in a general setting which may be characterized as follows. There is a set of experts $E = \{e_j\}_{j\in J}$ and a set of options (alternatives) $O = \{o_i\}_{i\in I}$. Each expert e_j is assumed to represent his or her testimonies over the set of options O in the form of a binary preference relation $R_j \subset O \times O$ where $R_j(o_{i_1}, o_{i_2})$ is meant to represent preference of the expert e_j for the option o_{i_1} over the option o_{i_2}, i.e., that in his or her opinion option o_{i_1} is better than the option o_{i_2}. Preference relations R_j may be assumed to exhibit various properties. Often, the transitivity ($R_j(o_{i_1}, o_{i_2}) \wedge R_j(o_{i_2}, o_{i_3})$ implies $R_j(o_{i_1}, o_{i_3})$), completeness ($\forall i_1, i_2 \in I$ either $R_j(o_1, o_2)$ or $R_j(o_2, o_1)$ holds), and some form of anti-symmetry (e.g., $R_j(o_1, o_2) \implies \neg R_j(o_2, o_1)$), is assumed.

In such a setting, the *social choice function* F may be defined as follows:

$$F(X, \{R_j\}) = Y \tag{1}$$

where $X, Y \subseteq O$ are sets of options such that $Y \subseteq X$, and $\{R_j\}$ is a set of preference relations on O.

Thus, a social choice function determines which options Y are to be selected from a set of options X in view of the preference relations $\{R_j\}$ of a group of experts.

Voting procedures used in the *elections* may be interpreted as social choice functions. Often, the voting procedure is required to indicate as Y exactly one element subset of X (cf. (1)), i.e., $Y = \{o_i\}$ and the option (candidate) o_i is then called the *winner* of the election. In case of voting procedures we will usually refer to experts and options as voters and candidates, respectively.

Particular voting procedures differ in that how the winner is selected. For example, some arrive at the decision in an iterative way and the voters are requested to express their preferences several times, often with respect to a changing set of candidates. Often, the *agenda* is established which determines in which order the candidates are voted for. Most of voting procedures do not require voters to express their whole preference relations, at least not at the very beginning, but assuming existence of such a complete preference relation (*ranking* of the candidates) makes it possible to derive the winner of the election (assuming they always vote in accordance with their complete preference relation).

There are many postulated properties which are desired to be met by a fair voting procedure properly reflecting the preferences of the voters. However, it turns out that it is impossible to find one possessing all desired properties. Thus, satisfaction of such desired properties may be treated as criteria in evaluation of particular voting procedures.

In our approach, based on our previous work [3], [11], [13], [23], our point of departure is the following list of desired properties (criteria) of voting procedures:

A - Condorcet winner If each time a candidate is preferred by the majority of voters when compared to any other candidate then it has to be the winner.

B - Condorcet loser If all other candidates are preferred by the majority of voters when compared to a given candidate then the latter candidate cannot be the winner.

C - majority winner if a candidate is top-ranked in the rankings of the majority (more than 50%) of voters then this candidate have to be the winner.

D - monotonicity If a candidate is the winner then if it is ranked higher by a voter then it has still to be the winner and if a candidate is not the winner then if ranked lower by a voter cannot become the winner.

E - weak Pareto winner If for a given candidate o_1 there exists another candidate o_2 which is ranked higher than o_1 by all voters then o_1 cannot be the winner.

F - consistency If the set of voters E is divided in two groups ($E = E_1 \cup E_2$), in any possible way, and a candidate is the winner both for E_1 and E_2 then it has to be the winner for E.

G - heritage If a candidate $o_i \in O$ is the winner then it has to be the winner also when any subset of candidates $O_1 \subseteq O$ is considered such that $o_i \in O_1$

In the paper we will consider some popular voting procedures which are briefly characterized below.

Amendment Candidates are voted individually, in some order, and if a candidate gets the majority of votes it becomes the winner; otherwise the next candidate is voted.

Copeland the winner is a candidate for which the highest is the difference between the numbers of pairwise comparisons with other candidates in which it is voted by majority, respectively, as better and as worse.

Dodgson the winner is the candidate for which the minimum number of changes in voters rankings is needed to make it a Condorcet winner.

Schwartz if there is a Condorcet winner it is the winner; otherwise the set $O_S \subseteq O$ of all candidates who are voted as better by majority of voters in pairwise comparison with all candidates belonging to the set $O \setminus O_S$ are the winners.

Max-min/Egalitarian The winner is the candidate whose worst position over the rankings of all voters is the highest.

Plurality Only top-ranked candidates for each voter are taken into account and the winner is the one which is most often among them.

Borda Each position in the ranking is assigned a score, highest for the top position and lowest for the last one and the winner is a candidate for which the sum of scores of the positions it takes in rankings of particular voters (the Borda count) is the highest.

Approval Each voter points out a subset of preferred candidates and the winner is the option which is present in the highest numer of these subsets.

Black The winner is the Condorcet winner, if it exists; otherwise the Borda voting procedure is used.

Runoff Works like Plurality but two best candidates are selected and then Plurality voting is repeated for just two of them.

Nanson The Borda voting procedure is iteratively repeated and in each iteration a candidate with the lowest Borda count is excluded from the voting in the following iteration.

Hare The Plurality voting procedure is iteratively repeated and in each iteration candidates with the lowest number of top positions in the rankings are excluded from the voting in the following iteration.

Coombs The winner is a candidate which is top-ranked by the majority of voters, if it exists. Otherwise, the procedure is iteratively repeated but in each iteration the candidate which is most often ranked as the last one is eliminated.

III. Formal Concept Analysis

Formal concept analysis (FCA) was introduced by Wille in [32]. FCA is founded on lattice theory and aimed at data analysis and representation. FCA uses tabular-type data representations called *formal contexts* where objects are characterized by mono-valued attributes[1]. In FCA data are represented and analyzed by concept lattices using algebraic, order and logical methods based on concept lattices. A construction of concept lattices is based on Galois connections determined by formal contexts. Here we present basic notions of FCA. For a detailed

[1]In the process of development, FCA was broadened also for multi-valued attributes by means of conceptual scaling [6].

presentation of formal concept analysis see the first monograph on FCA by Ganter and Wille [6] and for elements of lattice theory see an excellent textbook freely available on-line by Burris and Sankappanavar [2].

A *formal context* is defined as a triple of the form (G, M, I), where G and M are sets, while I is a binary relation $I \subseteq G \times M$. Elements of set G and M are called *objects* and *attributes* respectively as well as *an extent* and *an intent*, respectively, of the context (G, M, I). The fact that $a \in G$ and $m \in M$ are in relation I will be denoted as $a\ I\ m$, and will be described as that *object a possesses attribute m*, or that *attribute m is possessed by object a*.

For context (G, M, I) two different operators between power sets $\wp(G)$ and $\wp(M)$ are defined:

$$X \mapsto X^i = \{m \in M : a\ I\ m,\ \forall\ a \in X\},$$

$$Y \mapsto Y^e = \{a \in G : a\ I\ m,\ \forall\ m \in Y\},$$

for each $X \subseteq G$, $Y \subseteq M$. Operator $\cdot^i$ is called *an intension operator* and operator $\cdot^e$ is called *an extension operator*. One can note that operators $\cdot^i$ and $\cdot^e$ are perfectly dual in the sense of order theory, thus in FCA there is commonly used practice to denote these operators by the same prime symbol $\cdot'$ [6]. This practice is justified by the formal properties of extension and intension operators presented in Table I and makes calculation easier. It also does not lead into confusion: in Table I for example, since $Y \subseteq M$, then formula (3b) $Y' = Y'''$ can be rewritten as $Y^e = Y^{eie}$.

TABLE I
BASIC PROPERTIES OF INTENSION AND EXTENSION OPERATORS FOR FORMAL CONTEXT (G, M, I) AND SETS $X, X_1, X_2 \subseteq G$ AND $Y, Y_1, Y_2 \subseteq M$ [6].

(1a) $X_1 \subseteq X_2 \Rightarrow X_2' \subseteq X_1'$	(1b) $Y_1 \subseteq Y_2 \Rightarrow Y_2' \subseteq Y_1'$
(2a) $X \subseteq X''$	(2b) $Y \subseteq Y''$
(3a) $X' = X'''$	(3b) $Y' = Y'''$
(4a) $(X_1 \cup X_2)' = X_1' \cap X_2'$	(4b) $(Y_1 \cup Y_2)' = Y_1' \cap Y_2'$

A *formal concept* of context (G, M, I) is pair (A, B) with $A \subseteq G$, $B \subseteq M$, such that $A = B'$ and $B = A'$. A and B are called *the extent* and *the intent* of the concept (A, B) respectively. The family of all formal concepts of context (G, M, I) is denoted by $\mathfrak{B}(G, M, I)$. If $(A, B) \in \mathfrak{B}(G, M, I)$ and $g \in A$, then g is *an object from the concept* (A, B). Using properties form Table I one can show that for any object $g \in G$ and any attribute $m \in M$, the following equations hold: $(\{g\}'', \{g\}'), (\{m\}', \{m\}'') \in \mathfrak{B}(G, M, I)$. Concept $(\{g\}'', \{g\}')$ is called *an object concept of object g* whereas concept $(\{m\}', \{m\}'')$ is *an attribute concept of attribute m*. The object concept of any object $g \in G$ we denote by $\tilde{\gamma}(g)$ and the attribute concept of any attribute $m \in M$ we denote by $\tilde{\mu}(m)$. If $(A, B) = (\{g\}'', \{g\}')$, then object g is called *an own object* of concept (A, B), i.e. g posses only those attributes which are contained in B. Analogically, If $(A, B) = (\{m\}', \{m\}'')$, then attribute m is called *an own attribute of concept* (A, B), i.e. m is possessed only by those objects which are contained in A.

Let (G, M, I) be a formal context. On family $\mathfrak{B}(G, M, I)$ we define relation $\preccurlyeq$ in the following way:

$$(A_1, B_1) \preccurlyeq (A_2, B_2) \Leftrightarrow A_1 \subseteq A_2 (:\Leftrightarrow B_2 \subseteq B_1).$$

where $(A_1, B_1), (A_2, B_2) \in \mathfrak{B}(G, M, I)$. In this case (A_1, B_1) is called *a subconcept* of (A_2, B_2) and (A_2, B_2) is called *a superconcept* of (A_1, B_1). The relation $\preccurlyeq$ is a partial order on the family $\mathfrak{B}(G, M, I)$ and it is called *the hierarchical order* (or simply *order*). One can show that the family $\mathfrak{B}(G, M, I)$ ordered by the relation $\preccurlyeq$ is a complete lattice called the *concept lattice* of the context (G, M, I). We denote that lattice by $\underline{\mathfrak{B}}(G, M, I)$.

Having the relation $\preccurlyeq$ defined for the concept lattice one can equivalently consider two binary operations denoted as $\wedge$ and $\vee$ which can be expressed in terms, respectively, of the *infimum* and *supremum* with respect to relation $\preccurlyeq$. Namely, $a \wedge b = \inf\{a, b\}$ and $a \vee b = \sup\{a, b\}$. Thanks to the semantics of the infimum and supremum, these operations may be easily extended for arbitrary sets of arguments. The Basic Theorem on Concept Lattices [32], [6] shows that in the case of the concept lattice $\underline{\mathfrak{B}}(G, M, I)$ these operations and their generalizations for arbitrary sets of concepts are given by the following equations respectively:

$$\bigwedge_{i \in I}(A_i, B_i) = (\bigcap_{i \in I} A_i, (\bigcup_{i \in I} B_i)''),$$

$$\bigvee_{i \in I}(A_i, B_i) = ((\bigcup_{i \in I} A_i)'', \bigcap_{i \in I} B_i).$$

Therefore concept lattices can be viewed as hierarchical conceptual structures equipped with some operations on concepts and representing data stored in formal contexts. When the number of objects or the number of attributes in formal contexts are relatively small, then concept lattices can be used also for visualization of information stored in their formal contexts. In the next section we present a relatively small concept lattice representing a selection of social choice functions characterized by selected voting criteria, briefly introduced in section II.

IV. Concept Lattice of Social Choice Functions

This section is devoted to construction and structural analysis of proposed lattice of social choice functions. We start with definition of formal context of social choice functions on the basis of consideration conducted in the previous section. Let

$$\mathbb{SCF} := (\mathrm{G}_{\mathbb{SCF}}, \mathrm{M}_{\mathbb{SCF}}, \mathrm{I}_{\mathbb{SCF}})$$

be a formal context where set $\mathrm{G}_{\mathbb{SCF}}$ comprises all voting procedures presented in Section II, set $\mathrm{M}_{\mathbb{SCF}}$ consists of selected criteria denoted by letters $A, ..., G$ in Section II, while set of pairs $\mathrm{I}_{\mathbb{SCF}}$ is the incidence relation presented in Table II.

Now, on the basis of formal context $\mathbb{SCF}$ we construct the concept lattice of social choice functions $\underline{\mathfrak{B}}(\mathrm{G}_{\mathbb{SCF}}, \mathrm{M}_{\mathbb{SCF}}, \mathrm{I}_{\mathbb{SCF}})$ (Fig. 1) and study its properties[2].

[2]Diagrams of concept lattices are generated with usage of ConExp software by Serhiy A. Yevtushenko.

TABLE II
A FORMAL CONTEXT OF SELECTED SOCIAL CHOICE FUNCTIONS. ROWS CORRESPOND TO FORMAL OBJECTS WHICH ARE SOCIAL CHOICE FUNCTIONS AND COLUMNS CORRESPOND TO FORMAL ATTRIBUTES WHICH ARE SOME CRITERIA INTRODUCED IN SECTION II.

Voting procedures	Criteria						
	A	B	C	D	E	F	G
Amendment	×	×	×	×			
Copeland	×	×	×	×	×		
Dodgson	×		×		×		
Schwartz	×	×	×	×			
Max-min	×		×	×	×		
Plurality			×	×	×	×	
Borda		×		×	×	×	
Approval				×		×	×
Black	×	×	×	×	×		
Runoff		×	×		×		
Nanson	×	×	×		×		
Hare		×	×		×		
Coombs		×	×		×		

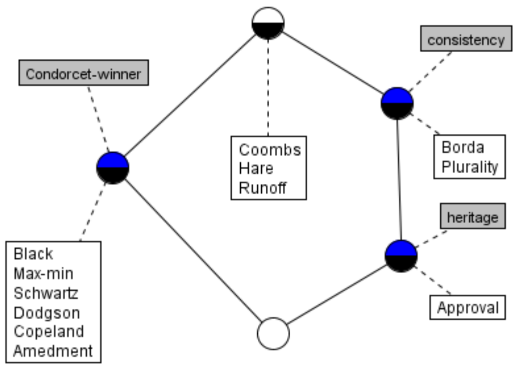

Fig. 2. Concept lattice $\underline{\mathfrak{B}}(\mathrm{G}_1, \mathrm{M}_1, \mathrm{I}_1)$.

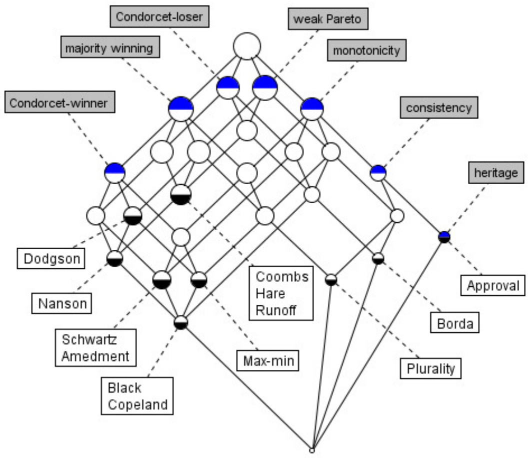

Fig. 1. Concept lattice of social choice functions $\underline{\mathfrak{B}}(\mathrm{G}_{\mathbb{SCF}}, \mathrm{M}_{\mathbb{SCF}}, \mathrm{I}_{\mathbb{SCF}})$. Half-black nodes represent object concepts while half-blue nodes represent attribute concepts .

Fact 1: Concept lattice of social choice functions $\underline{\mathfrak{B}}(\mathrm{G}_{\mathbb{SCF}}, \mathrm{M}_{\mathbb{SCF}}, \mathrm{I}_{\mathbb{SCF}})$ is nondistributive, i.e., it is not the case that $\forall C_1, C_2, C_3 \in \mathfrak{B}(\mathrm{G}_{\mathbb{SCF}}, \mathrm{M}_{\mathbb{SCF}}, \mathrm{I}_{\mathbb{SCF}})$ $C_1 \wedge (C_2 \vee C_3) = (C_1 \wedge C_2) \vee (C_1 \wedge C_3)$ nor $C_1 \vee (C_2 \wedge C_3) = (C_1 \vee C_2) \wedge (C_1 \vee C_3)$.

In order to show this one can consider the following formal context:

$$\mathbb{K}_1 := (\mathrm{G}_1, \mathrm{M}_1, \mathrm{I}_1),$$

where $\mathrm{G}_1 := \mathrm{G}_{\mathbb{SCF}}$, $\mathrm{M}_1 := \{Condorcet-winner,\ consistency,\ heritage\}$, and $\mathrm{I}_1 := \mathrm{I}_{\mathbb{SCF}} \cap (G_1 \times M_1)$. Now one can note that its concept lattice $\underline{\mathfrak{B}}(\mathrm{G}_1, \mathrm{M}_1, \mathrm{I}_1)$, which is presented in Fig. 2, is a famous N5 lattice [2] and can be embedded into concept lattice of social choice functions $\underline{\mathfrak{B}}(\mathrm{G}_{\mathbb{SCF}}, \mathrm{M}_{\mathbb{SCF}}, \mathrm{I}_{\mathbb{SCF}})$. This by famous theorems by Dedekind and by Birkhoff implies that concept lattice of social choice functions $\underline{\mathfrak{B}}(\mathrm{G}_{\mathbb{SCF}}, \mathrm{M}_{\mathbb{SCF}}, \mathrm{I}_{\mathbb{SCF}})$ is neither modular nor distributive (see e.g. [2]).

The fact that concept lattice $\underline{\mathfrak{B}}(\mathrm{G}_{\mathbb{SCF}}, \mathrm{M}_{\mathbb{SCF}}, \mathrm{I}_{\mathbb{SCF}})$ lacks such regular property as distibutivity is not surprising. In fact it is quite rare phenomenon that real, non-manipulated data generate concept lattice possessing some regular properties. For example, one can consult concept lattice presented in [17] and generated from Threats Matrix (in German *Gefahrenmatrix*) used in commanding of tactical actions by German Fire Service [1], [7]. However, looking at the concept lattice presented in Fig. 1 one can note that the left part of this lattice diagram reveals some regularity. Namely, the concept lattice of social choice functions $\underline{\mathfrak{B}}(\mathrm{G}_1, \mathrm{M}_1, \mathrm{I}_1)$ contains as sublattices some distributive lattices or some Boolean algebras. Moreover, some subcontexts generated from context $\mathbb{SCF}$ of social choice functions generate lattices possessing some regularity.

Let

$$\mathbb{K}_2 := (\mathrm{G}_2, \mathrm{M}_2, \mathrm{I}_2),$$

be a formal context, where $\mathrm{G}_2 := \mathrm{G}_{\mathbb{SCF}}$, $\mathrm{M}_2 := \{Condorcet-winner, majority\ winning,\ weak\ Pareto,\ monotonicity\}$, and $\mathrm{I}_2 := \mathrm{I}_{\mathbb{SCF}} \cap (G_2 \times M_2)$, thus $\mathbb{K}_2$ is subcontext of social choice functions context $\mathbb{SCF}$. Then concept lattice $\underline{\mathfrak{B}}(\mathrm{G}_2, \mathrm{M}_2, \mathrm{I}_2)$ is presented in Figure 3. One can note that concept lattice $\underline{\mathfrak{B}}(\mathbb{K}_2)$ is distributive.

For another example let us consider the following subcontext of social choice functions context $\mathbb{SCF}$:

$$\mathbb{K}_3 := (\mathrm{G}_3, \mathrm{M}_3, \mathrm{I}_3),$$

where $\mathrm{G}_3 := \mathrm{G}_{\mathbb{SCF}}$, $\mathrm{M}_3 := \{Condorcet-loser,\ majority\ winning,\ weak\ Pareto, monotonicity\}$, and $\mathrm{I}_3 := \mathrm{I}_{\mathbb{SCF}} \cap (G_3 \times M_3)$. One can note that concept lattice $\underline{\mathfrak{B}}(\mathbb{K}_3)$ presented in Figure 4 is Boolean lattice isomorphic to the power set algebra of a four-element set.

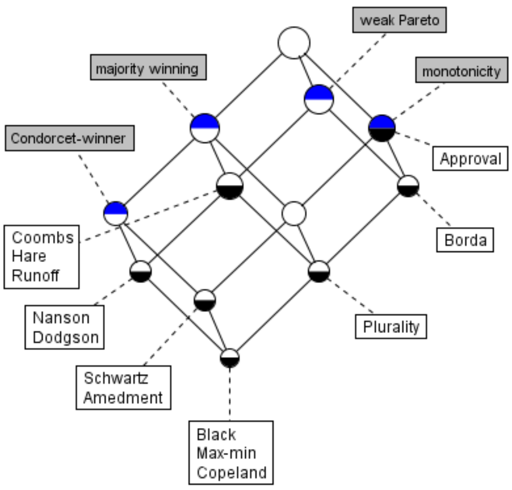

Fig. 3. Distributive concept lattice $\underline{\mathfrak{B}}(\mathrm{G}_2, \mathrm{M}_2, \mathrm{I}_2)$.

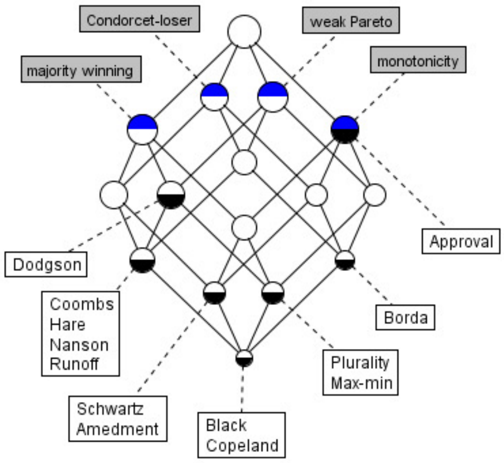

Fig. 4. Boolean concept lattice $\underline{\mathfrak{B}}(\mathrm{G}_3, \mathrm{M}_3, \mathrm{I}_3)$.

V. Information Provided by Lattice of Social Choice Functions

This section is devoted to the analysis of social choice functions listed in Section II by means of the FCA. We will concentrate on issues of attribute independency, reduction of information and on attribute implications derived from proposed formal context of social choice functions by means of FCA.

A. Attribute Independency

Let us start our consideration with the independency of attributes in the formal context $\mathbb{SCF}$ of social choice functions (voting procedures). Let (G, M, I) be the formal context. Attributes in $X \subseteq M$ are independent if there are no trivial dependencies between them i.e. functional (or ordinal) dependencies where set of attributes Y is functionally (ordinally) dependent on set of attributes X and $Y \subseteq X$. Following [29] we recall:

Lemma 1: Attributes are independent if they span a hypercube in a concept lattice.

For example concept lattice of social choice functions has four coatoms and these formal concepts as coatoms are also attribute concepts, namely these are $\tilde{\mu}(majority\ winning)$, $\tilde{\mu}(Condorcet - loser)$, $\tilde{\mu}(weak\ Paretto)$ and $\tilde{\mu}(monotonicity)$. These attributes are independent since every three attributes from this set (by their attribute concepts) span a hypercube in the concept lattice $\underline{\mathfrak{B}}(\mathrm{G}_{\mathbb{SCF}}, \mathrm{M}_{\mathbb{SCF}}, \mathrm{I}_{\mathbb{SCF}})$. In fact, in concept lattice $\underline{\mathfrak{B}}(\mathrm{G}_{\mathbb{SCF}}, \mathrm{M}_{\mathbb{SCF}}, \mathrm{I}_{\mathbb{SCF}})$ it is easier to characterize sets of attributes which are not independent: in concept lattice $\underline{\mathfrak{B}}(\mathbb{SCF})$ there are four two-element chains of attribute concepts, namely:

- $\{\tilde{\mu}(Condorcet - winner), \tilde{\mu}(majority\ winning)\}$,
- $\{\tilde{\mu}(heritage),\ \tilde{\mu}(consistency)\}$,
- $\{\tilde{\mu}(heritage),\ \tilde{\mu}(monotonicity)\}$,
- $\{\tilde{\mu}(consistency),\ \tilde{\mu}(monotonicity)\}$.

Sets of attributes which are not independent are exactly sets of attributes containing at least one pair of attributes such that their attribute concepts are contained in one of the above two-element chains.

B. Reducibility of Information

Reduction of information is one of the main advantages of formal concept analysis. Here we describe reduction of information within concept lattice of social choice functions.

Let us recall that context (G, M, I) is called *clarified* if for any objects $g, h \in G$, $g' = h'$ implies $g = h$ and for any attributes $m, n \in M$, $m' = n'$ implies $m = n$. Now let us note the following facts:

Fact 2: Context $\mathbb{SCF}$ is not clarified.

It is so since, e.g., $\{Coombs\}' = \{Runoff\}'$ but obviously $Coombs \neq Runoff$. However, for the set of all attributes of context $\mathbb{SCF}$ (denoted by $\mathrm{M}_{\mathbb{SCF}}$) one of the necessary conditions for a clarified context holds, i.e.:

Fact 3: For all attributes (criteria) $m, n \in \mathrm{M}_{\mathbb{SCF}}$ the following implication holds:

$$\{m\}' = \{n\}' \Rightarrow m = n.$$

It is easily seen in Fig. 1 where there are no two criteria determining the same attribute concept.

Let us recall that for any formal context (G, M, I), object $g \in G$ is *reducible* if its object concept $\tilde{\gamma}(g)$is *supremum-reducible*, i.e., can be represented as the supremum of strictly smaller concepts what implies that concept $\tilde{\gamma}(g)$ has no unique lower neighbour in concept lattice $\underline{\mathfrak{B}}(G,M,I)$. Analogously, for any context (G,M,I), attribute $m \in M$ is *reducible* if its attribute concept $\tilde{\mu}(m)$ is *infimum-reducible*, i.e., can be represented as infimum of strictly greater concepts, i.e. concept $\tilde{\gamma}(m)$ has no unique upper neighbour in concept lattice $\underline{\mathfrak{B}}(G,M,I)$. Now one can note that:

Fact 4: The social choice function Dodgson (more formally: the object representing this social function in the lattice) is reducible. The rest of social choice functions from the context $\mathbb{SCF}$ are irreducible.

One can note that social choice function $Dodgson$ as a formal object is reducible since its object concept is a supremum of two different concepts:

$$\tilde{\gamma}(Dodgson) = \tilde{\gamma}(Nanson) \vee \tilde{\gamma}(Max - min),$$

in $\underline{\mathfrak{B}}(G,M,I)$, the concept lattice of social choice functions. Namely the object concept $\tilde{\gamma}(Dodgson)$ is the lattice union of the object concepts determined by voting procedures Nanson and the object concept determined by the voting procedure Max-min. This observation may be expressed in a different way by saying that the Dodgson social choice function is Pareto-dominated by the Nanson and Max-min functions. Such a statement is justified as the attributes of the functions express their desired properties and thus the aforementioned dominance is here well-defined.

Concerning the rest of voting procedures from the social choice functions context $\mathbb{SCF}$, their object concepts have exactly one lower neighbour in the concept lattice of social choice functions, thus by Proposition 2 of [6] these object concepts are irreducible.

Fact 5: All attributes are irreducible.

One can note that every attribute concept determined by a criterion from the context $\mathbb{SCF}$ has exactly one upper neighbour what in the light of Proposition 2 of [6] shows that all attribute concepts in the social choice context $\mathbb{SCF}$ are infimum-irreducible.

Fact 6: In the formal context $\mathbb{SCF}$ of social choice functions there is only one concept which is both object concept and attribute concept.

In order to show this one can note that:

$$\tilde{\gamma}(Approval) = \tilde{\mu}(heritage),$$

i.e. social choice function (voting procedure)$Approval$ and voting criterion $heritage$ determine the same concept in the concept lattice of social choice functions. It stems from the fact, that the property (attribute) $heritage$ distinguishes the voting procedure $Approval$ from the other procedures and, at the same time, property $heritage$ is satisfied only by $Approval$.

C. Implications holding in the Context of Social Choice Functions

The FCA based analysis of voting procedures brings in another potentially interesting insight into their functioning. Namely, *implications* holding in the context of social choice functions may provide social choice theorists with valuable information. Those implications are not laws derived by theoretical considerations directly from knowledge gathered in the framework of the social choice theory but they are derived from the description of the voting procedures created by politicians and social choice theorist and expressed in terms of different properties postulated by social choice theorists.

Let us recall the notion of attribute implication. Informally, implications between attributes are the statements of the following form "Every object with the attributes a, b, c, ... also has the attributes x, y, z, ... " [6]. Formally speaking, an implication between attributes in context (G,M,I) is a pair of subsets of the attribute set M. If $A,B \subseteq M$, then implication between A and B is denoted by $A \rightarrow B$. An implication between attributes may or may not hold in a given formal context. Instead of formal definition of implication which holds in a given formal context we recall a transparent characterization of this notion given in Proposition 19 in [6]: an implication $A \rightarrow B$ holds in (G,M,I) if and only if $B \subseteq A''$.

Looking at concept lattice of social choice functions presented in Fig. 1 one can note relatively large number of nontrivial implications between singular attributes which are enlisted below:

- $\{Condorcet - winner\} \rightarrow \{majority\ winning\}$
- $\{heritage\} \rightarrow \{consistency\}$
- $\{consistency\} \rightarrow \{monotonicity\}$

One of the formal reasons for that is the fact that five of seven attribute concepts are involved into two chains maximal with respect to the property that they consist only of attribute concepts, namely the following two chains:

- $\{\tilde{\mu}(Condorcet - winner), \tilde{\mu}(majority\ winning)\}$
- $\{\tilde{\mu}(heritage),\ \tilde{\mu}(consistency),\ \tilde{\mu}(monotonicity)\}$.

Finally, one can note that maximal antichains consisting only of attribute concepts have four elements which seems to be a relatively high number compared to the fact that maximal antichains in concept lattice of social choice functions analyzed within this paper have seven elements, i.e. the width of the concept lattice of social choice functions is 6.

VI. Conclusions

The concept lattice of social choice functions constructed and analyzed within this paper has an interesting and pretty regular structure. Despite the fact that itself it is nondistributive lattice it contains quite a few regular sublattices, including Boolean, distributive and modular lattices of a quite large size compared to the size of the whole lattice.

From the perspective of social choice theory interesting is a comparison of the applicability of formal concept analysis methods and rough sets theory methods. The latter has been already reported in the literature [3], [13]. One of particular dimensions of such comparison will be the issue of information reduction in both approaches.

And last but not least, an interesting issue worth of further research from the perspective of FCA is to find out whether observations reported in the paper can be interpreted in a deeper way in the language of the social choice theory. Thus, the further research in this direction can be focused on one task: to understand the observed phenomena presented in this paper in terms of social choice theory.

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