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A lower bound for proportion of visibility polygon's surface to entire polygon's surface: Estimated by Art Gallery Problem and proven that cannot be greatly improved

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***Abstract*—Assuming a bounded polygon and a point inside the polygon or on its boundary, the visibility polygon, also called the visibility region, is a polygon reachable, i.e., visible by straight lines from the point without hitting the polygon's edges or vertices. If the polygon is bounded, then the visibility polygon is bounded, and the proportion of the visibility polygon's surface area to the given polygon's surface area could be enumerated. Many papers investigate applications of the visibility polygons in robotics and computer graphics or focus on computationally effective finding the visibility region for a given polygon. However, surprisingly, there seems to be no work estimating the proportion of a visibility polygon's surface to an entire polygon's surface or its bounds. Thus, in this paper, we search for a lower bound of the surface proportion of a visibility polygon to a given one. Assuming n-sided simple polygon, i.e., a polygon without holes and edge intersections, we apply the well-known art gallery problem and derive there is always a point inside the polygon or on its boundary that guarantees the proportion of the visibility polygon's surface to the entire polygon's surface is at least $\frac{1}{\lfloor n/3 \rfloor}$. We also show that there are n-sided polygons for which the proportion of the visibility polygon's surface to the entire polygon's surface is asymptotically not greater than $\frac{1}{\lfloor n/3 \rfloor}$ for any point inside the polygon or on its boundary. So, the lower bound of the proportion of the visibility polygon's surface to the entire polygon's surface, $\frac{1}{\lfloor n/3 \rfloor}$, cannot be improved in general.**

I. Introduction

THE visibility region of a polygon related to a given polygon's point, i.e., the largest part of the polygon, so that each point of such a polygon's part is directly visible from the given point, has multiple applications in robotics, operational research, and logistics, security, video game creation, and other situations, mainly of optimization character.

In robotics, visibility regions (polygons) are typically important for robotic agents to enable appropriate movement-making and planning [1]. There is a well-known problem called facility location problem where a number of facilities are to be optimally placed to minimize any transportation costs [2]. Besides other approaches, such as clustering, the visibility polygons could help to find the optimal facility setting [3], [4]. Similarly, in security applications, areas of various geometric shapes are often required to be guarded – then, a number and placement of guards watching the area could be researched using visibility polygons [5].

Since most problems are based on specific n-sided polygons, many papers search for an algorithm for building the visibility polygon in the shortest possible asymptotic time complexity. The naive approach takes quadratic time in terms of n – each pair of every two vertices is inspected to determine whether they are visible from a given point. While Asano published a faster sweeping algorithm for the visibility polygon construction, taking $n \log n$ asymptotic time [6], Lee developed the algorithm in linear time [7] by tricky stacking. Afterward, Joe and Simpson made Lee's algorithm even more robust, keeping it still in linear time [8].

In this paper, we go deeper rather into an estimate of a proportion of the visibility region's surface to the polygon's surface, assuming a point inside the polygon or on its boundary. In general, publications on this topic are missing. A guaranteed lower bound of the proportion, if this would be sufficiently high for at least one polygon's point, could, for instance, help in various tasks to decide whether one point with its visibility region is enough to satisfy the task conditions. On the other hand, cases of polygons that would show the lower bound of the proportion could not be greater than, e.g., some constant, might also imply that there is, for example, no optimal solution of the given task using only one chosen point. We research both situations more in detail and apply the outcomes on the *garden-watering problem*.

II. Preliminaries

Firstly, let us define the visibility region and the proportion of a visibility polygon's surface to an entire polygon's surface. In general, we do not assume convexity of polygons.

Definition 1 (The visibility polygon, visibility region)**.** *Let us assume a simple non-empty n-sided polygon $\mathcal{P}$, i.e., a polygon that does not intersect itself and has no holes, with $n \in \mathbb{N}$ and $n \geq 3$. Let A be a point inside polygon $\mathcal{P}$ or on its boundary, i.e., $A \in \mathcal{P}$. A visibility polygon (region) for point A in polygon $\mathcal{P}$ is polygon $\mathcal{V}_{A,\mathcal{P}}$ created by a set of all points $B \in \mathcal{P}$ so that each segment AB lies completely in $\mathcal{P}$, i.e., $AB \in \mathcal{P}$.*

As an example, the visibility polygon (region) of the polygon in Fig. 1 is colored in gray.

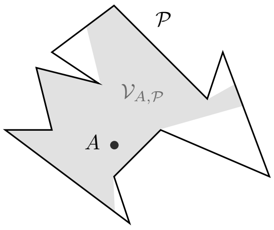

Fig. 1. An example of n-sided polygon $\mathcal{P}$ with $n = 12$. The visibility polygon $\mathcal{V}_{A,\mathcal{P}}$ for point A in polygon $\mathcal{P}$ is in gray color.

Definition 2 (A proportion of a visibility polygon's surface to an entire polygon's surface)**.** *Following definition 1, let us assume $n \in \mathbb{N}$ so that $n \geq 3$, a simple non-empty n-sided polygon $\mathcal{P}$ and a point $A \in \mathcal{P}$. Let $S(\mathcal{V}_{A,\mathcal{P}})$ be a surface of the visibility polygon for point A and $S(\mathcal{P})$ be a surface of polygon $\mathcal{P}$. Then, the proportion of the visibility polygon's surface $S(\mathcal{V}_{A,\mathcal{P}})$ to the entire polygon's surface $S(\mathcal{P})$ is marked ν and is equal to*

$$\nu = \frac{S(\mathcal{V}_{A,\mathcal{P}})}{S(\mathcal{P})}.$$

While trivial upper and lower bounds of the proportion of a visibility polygon's surface to an entire polygon's surface are apparent, as shown in the following lemma, more efficient estimates of the proportion bounds might be tricky, though.

Lemma 1. *Assuming definition 2, the proportion of a visibility polygon's surface $S(\mathcal{V}_{A,\mathcal{P}})$ to an entire polygon's surface $S(\mathcal{P})$ is always*

$$0 \leq \nu = \frac{S(\mathcal{V}_{A,\mathcal{P}})}{S(\mathcal{P})} \leq 1.$$

Proof. The n-sided polygon $\mathcal{P}$ is non-empty, so $S(\mathcal{P}) > 0$. Since surely $S(\mathcal{V}_{A,\mathcal{P}}) \geq 0$, it is $\nu = \frac{S(\mathcal{V}_{A,\mathcal{P}})}{S(\mathcal{P})} \geq 0$. The visibility polygon $\mathcal{V}_{A,\mathcal{P}}$ is created by points $B \in \mathcal{P}$, thus $S(\mathcal{V}_{A,\mathcal{P}}) \leq S(\mathcal{P})$ and $\nu = \frac{S(\mathcal{V}_{A,\mathcal{P}})}{S(\mathcal{P})} \leq 1$. □

Due to the optimization fashion of the tasks using the visibility polygon, it is much more useful to investigate a lower bound of the proportion of a visibility polygon's surface to an entire polygon's surface, which may ensure that for a smart choice of the polygon's point, its visibility region is guaranteed to be sufficiently high. The proportion's upper bound equaled to 1, as shown in lemma 1, is often satisfied, e.g., for convex polygons, as proved below.

Lemma 2. *Assuming definition 2, the proportion of a visibility polygon's surface $S(\mathcal{V}_{A,\mathcal{P}})$ to an entire polygon's surface $S(\mathcal{P})$ is equal to 1 if polygon $\mathcal{P}$ is convex.*

Proof. Let us proof that if polygon $\mathcal{P}$ is convex, then for any point $A \in \mathcal{P}$ is $\mathcal{V}_{A,\mathcal{P}} = \mathcal{P}$. Polygon $\mathcal{P}$ is convex, so, for each points $C \in \mathcal{P}$ and $D \in \mathcal{P}$ holds that $CD \in \mathcal{P}$ (†). By contradiction, let's assume there is point $X \in \mathcal{P}$ so that $X \notin \mathcal{V}_{A,\mathcal{P}}$. From definition 1, since the visibility polygon $\mathcal{V}_{A,\mathcal{P}}$ is created by points $B \in \mathcal{P}$, it is surely $\mathcal{V}_{A,\mathcal{P}} \subseteq \mathcal{P}$. If $X \notin \mathcal{V}_{A,\mathcal{P}}$, then $AX \notin \mathcal{P}$, otherwise, due to definition 1, necessarily would be $X \in \mathcal{V}_{A,\mathcal{P}}$. But, if $AX \notin \mathcal{P}$ and both $A \in \mathcal{P}$ and $X \in \mathcal{P}$, this is contrary to (†). Thus, if polygon $\mathcal{P}$ is convex, then for any point $A \in \mathcal{P}$ is $\mathcal{V}_{A,\mathcal{P}} = \mathcal{P}$, so $S(\mathcal{V}_{A,\mathcal{P}}) = S(\mathcal{P}) > 0$ and $\nu = \frac{S(\mathcal{V}_{A,\mathcal{P}})}{S(\mathcal{P})} = 1$. □

If one could ensure that there always exists a point in a polygon so that the proportion of the surface of the visibility polygon for the point and surface of the entire polygon is greater than or equal to a constant, derivable from the polygon's characteristics, many of the optimization tasks, e.g., in security or logistics could have a plausible solution using only one point (or agent). In another way, there is a class of polygons for which the proportion of a visibility polygon's surface to an entire polygon's surface is not much greater than the constant from the previous case, regardless of the polygon's point choice; it is additional information for the task solution, too – there likely does not exist a satisfying solution using only one point (agent).

III. More on a lower bound of the proportion of a visibility polygon's surface to an entire polygon's surface

In the following sections, we investigate the existence of a simple n-sided polygon's point, i.e., a point in the polygon or on its boundary, so that a proportion of the surface of the visibility polygon for the point and the surface of the entire polygon is always greater than or equal to a constant, related to n. Also, we demonstrate there are n-sided polygons so that each visibility polygon is not much greater than the constant, regardless of the polygon's point selection. Thus, we show the surface proportion that is always greater than or equal to a constant for at least one point in the polygon could not be much more effective.

A. The existence of a polygon's point guaranteeing that the proportion of a visibility polygon's surface to an entire polygon's surface is not lower than a polygon-related constant

Using the popular art gallery problem [9], we prove that, assuming a simple non-empty n-sided polygon, there always exists a point in the polygon so that the proportion of the surface of the visibility polygon for the point, and the surface of the entire n-sided polygon is greater than or equal to $\frac{1}{\lfloor n/3 \rfloor}$.

Let us start with the art gallery problem, first introduced by Chvátal in [9].

Theorem 1 (The art gallery problem). *Assume $n \in \mathbb{N}$ so that $n \geq 3$ and an art gallery of a shape following a simple non-empty n-sided polygon $\mathcal{P}$. Then, $\lfloor n/3 \rfloor$ guards are enough to watch the entire area of the art gallery, i.e., each point in the art gallery's polygon is visible by at least one of the $\lfloor n/3 \rfloor$ guards.*

Proof. The first proof of the classical art gallery problem was published in [9], and the short and elegant one came from [10]. □

In Fig. 2, there is an illustration of the art gallery problem for an art gallery following a shape of n-sided polygon $\mathcal{P}$ with $n = 8$.

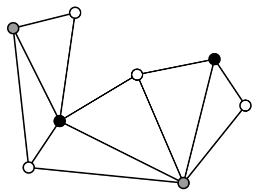

Fig. 2. An example of an art gallery following a shape of n-sided polygon $\mathcal{P}$ with $n = 8$. The n-sided polygon is triangulated, and white, gray, and black colors color the vertices of each triangle. As in theorem 1, $\lfloor n/3 \rfloor = \lfloor 8/3 \rfloor = 2$ guards, placed in vertices of black or gray color, is enough to ensure each point of the polygon is visible by at least one of them. (However, still, in fact, one guard, placed in the left bottom white vertex, is enough to watch the gallery.)

Working out some of the consequences of the art gallery problem, we get the following.

Theorem 2 (Surface of a visibility polygon of a guard in the art gallery). *Let us assume $n \in \mathbb{N}$ and $n \geq 3$, and an art gallery following a simple non-empty n-sided polygon $\mathcal{P}$ with surface $S(\mathcal{P})$. Also, let us assume there are $\lfloor n/3 \rfloor$ guards, placed in vertices of the polygon. There exists a guard, i.e., a point in the polygon or on its boundary, so that a surface of their visibility polygon is greater than or equal to $\frac{S(\mathcal{P})}{\lfloor n/3 \rfloor}$.*

Proof. Let us prove the theorem by contradiction. For $\forall i \in \{1, 2, \ldots, \lfloor n/3 \rfloor\}$, let $\mathcal{V}_{G_i,\mathcal{P}}$ be a visibility polygon of guard G_i and also let us assume that $S(\mathcal{V}_{G_i,\mathcal{P}}) < \frac{S(\mathcal{P})}{\lfloor n/3 \rfloor}$ (†). Due to theorem 1, $\lfloor n/3 \rfloor$ guards are enough to ensure that each point of the polygon is visible for one or more of them. Thus, the polygon created by union of all $\lfloor n/3 \rfloor$ guards' visibility polygons should be of a surface that covers the surface $S(\mathcal{P})$ of polygon $\mathcal{P}$. However, under (†), a surface of the union of all guards' visibility polygons is

$$
\begin{aligned}
S\left(\bigcup_{i=1}^{\lfloor n/3 \rfloor} \mathcal{V}_{G_i,\mathcal{P}}\right) &\leq \sum_{i=1}^{\lfloor n/3 \rfloor} S(\mathcal{V}_{G_i,\mathcal{P}}) \overset{(\dagger)}{<} \\
&\overset{(\dagger)}{<} \lfloor n/3 \rfloor \cdot \frac{S(\mathcal{P})}{\lfloor n/3 \rfloor} = \\
&= S(\mathcal{P}),
\end{aligned}
$$

which is contrary to theorem 1's output that $\lfloor n/3 \rfloor$ guards sufficiently secure the polygon. So, (†) cannot be true and the surface of each guard's visibility polygon cannot be lower than $\frac{S(\mathcal{P})}{\lfloor n/3 \rfloor}$. Thus, there must exist at least one guard, i.e., a point in the polygon or on its boundary, with the visibility polygon of surface greater than or equal to $\frac{S(\mathcal{P})}{\lfloor n/3 \rfloor}$. □

Finally, now we can prove the main idea of the section.

Theorem 3. *Let us assume $n \in \mathbb{N}$ and $n \geq 3$, and a simple non-empty n-sided polygon $\mathcal{P}$ with surface $S(\mathcal{P})$. There always exists a point A in polygon $\mathcal{P}$ or on its boundary, so that the proportion of surface $S(\mathcal{V}_{A,\mathcal{P}})$ of visibility polygon $\mathcal{V}_{A,\mathcal{P}}$ for the point A, and surface $S(\mathcal{P})$ of polygon $\mathcal{P}$ is greater than or equal to $\frac{1}{\lfloor n/3 \rfloor}$, i.e.,*

$$\nu = \frac{S(\mathcal{V}_{A,\mathcal{P}})}{S(\mathcal{P})} \geq \frac{1}{\lfloor n/3 \rfloor}.$$

Proof. Revisiting theorem 2, the point A is identical to the guard with a visibility polygon of surface at least $\frac{S(\mathcal{P})}{\lfloor n/3 \rfloor}$. The existence of such a guard is proved by theorem 1 and 2. Since the polygon is non-empty, i.e., $S(\mathcal{P}) > 0$ (‡), and surface of their visibility polygon is greater than or equal to $\frac{S(\mathcal{P})}{\lfloor n/3 \rfloor}$, it is $S(\mathcal{V}_{A,\mathcal{P}}) \geq \frac{S(\mathcal{P})}{\lfloor n/3 \rfloor}$ and

$$\nu = \frac{S(\mathcal{V}_{A,\mathcal{P}})}{S(\mathcal{P})} \geq \frac{\frac{S(\mathcal{P})}{\lfloor n/3 \rfloor}}{S(\mathcal{P})} \overset{(\ddagger)}{=} \frac{1}{\lfloor n/3 \rfloor}. \tag{1}$$

□

So, we demonstrated that there is always a point in a given simple n-sided polygon (convex or concave) that ensures that a proportion of its visibility polygon's surface and the entire polygon's surface is at least $\frac{1}{\lfloor n/3 \rfloor}$. Such knowledge could be handy in a class of tasks using the visibility polygon based on not necessarily coverage of a given polygon by the visibility polygon.

B. The polygons for which the proportion of a visibility polygon's surface to an entire polygon's surface is asymptotically not greater than a polygon-related constant

Although we showed that there must be a point in a simple n-sided polygon or on its boundary for which a proportion of its visibility polygon's surface and the entire polygon's surface is at least $\frac{1}{\lfloor n/3 \rfloor}$, for each $n \in \mathbb{N}$ where $n \geq 6$, there are polygons that regardless of the point choice, the surface proportion is asymptotically not greater than $\frac{1}{\lfloor n/3 \rfloor}$. Firstly, let's define such polygons and estimate the total surface of their visibility polygons.

Definition 3 (The saw-like polygons). *Let $n \in \mathbb{N}$ so that $n \geq 6$. A saw-like n-sided polygon is a concave polygon consisting of $k = \lfloor n/3 \rfloor$ periodically repeating triple structures: each structure includes a triangle part, a base part and a connection part. While the triangle part is an isosceles triangle with a base of length $a > 0$, the base and connection parts are rectangles with edges of length $a > 0$ and $\varepsilon > 0$, where $a \gg \varepsilon \gtrsim 0$. Thus, the triangle part's surface is much greater than the base or connection part's surface. If $n = 3k$ for $k \in \mathbb{N}$, then the rightmost connection part is missing, and* *if $n = 3k + 1$, then the rightmost connection part is halved into a triangle shape; see Fig. 3 for details and illustration.*

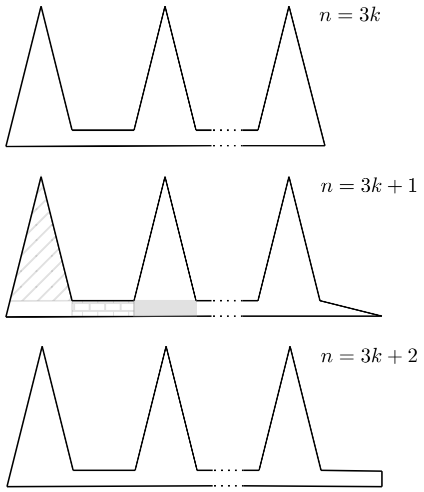

Fig. 3. Examples and patterns of the saw-like n-sided polygons. The triangle part is in a line hatch pattern, the base part is in gray, and the connection part is in a brick hatch pattern. Assuming $k \in \mathbb{N}$, then if $n = 3k$ for $k \in \mathbb{N}$, then the rightmost connection part is missing (top subfigure), if $n = 3k + 1$, the rightmost connection part is halved into a triangle shape (middle subfigure), if $n = 3k + 2$, the rightmost connection part is a rectangle as expected (bottom subfigure).

Lemma 3. *For each $n \in \mathbb{N}$ so that $n \geq 6$, i.e., $n = 3k$, or $n = 3k + 1$, or $n = 3k + 2$, where $k \in \mathbb{N}$, is $k = \lfloor n/3 \rfloor$.*

Proof. If $n = 3k$, then $k = \lfloor n/3 \rfloor = \left\lfloor \frac{3k}{3} \right\rfloor = \lfloor k \rfloor = k$. Else if $n = 3k + 1$, then $k = \lfloor n/3 \rfloor = \left\lfloor \frac{3k+1}{3} \right\rfloor = \left\lfloor k + \frac{1}{3} \right\rfloor = k$. Finally, if $n = 3k + 2$, then $k = \lfloor n/3 \rfloor = \left\lfloor \frac{3k+2}{3} \right\rfloor = \left\lfloor k + \frac{2}{3} \right\rfloor = k$. □

Theorem 4. *Let us assume a saw-like n-sided polygon $\mathcal{P}$ with $n \geq 6$ according to definition 3. For any point $A \in \mathcal{P}$, the proportion of its visibility polygon's surface $S(\mathcal{V}_{A,\mathcal{P}})$ to an entire polygon's surface $S(\mathcal{P})$ is asymptotically not greater than $\frac{1}{\lfloor n/3 \rfloor}$, i.e.,*

$$\nu = \frac{S(\mathcal{V}_{A,\mathcal{P}})}{S(\mathcal{P})} \lesssim \frac{1}{\lfloor n/3 \rfloor}.$$

Proof. Mark $S(\mathcal{T})$ a surface of the triangle part, $S(\mathcal{B})$ a surface of the base part, and $S(\mathcal{C})$ a surface of the connection part. Applying definition 3, obviously, it is $S(\mathcal{B}) = S(\mathcal{C}) = a\varepsilon$ ($\dagger$). Since lemma 3, polygon $\mathcal{P}$ contains exactly $k = \lfloor n/3 \rfloor$ structures ($\bullet$) consisting of one triangle, base and connection part (with exception for the rightmost connection part[1], see Fig. 3), the surface of polygon $\mathcal{P}$ is

$$S(\mathcal{P}) \geq k \cdot (S(\mathcal{T}) + S(\mathcal{B}) + S(\mathcal{C})) - S(\mathcal{C}). \tag{2}$$

For any point $A \in \mathcal{P}$ in any triangle part of polygon $\mathcal{P}$, the visibility polygon $\mathcal{V}_{A,\mathcal{P}}$ includes the triangle part and, at maximum, all base and connection parts; thus, its surface is

[1]Formula (2) could be precised according to whether $n = 3k$, $n = 3k + 1$ or $n = 3k + 2$; however, it has only a low significance for the proof.

$$S(\mathcal{V}_{A,\mathcal{P}}) \leq S(\mathcal{V}_{A,\mathcal{P}})_{\mathcal{T}} = S(\mathcal{T}) + k \cdot S(\mathcal{B}) + k \cdot S(\mathcal{C}), \tag{3}$$

see Fig. 4 for details. Also, for any point $A \in \mathcal{P}$ in any base part of polygon $\mathcal{P}$, the visibility polygon $\mathcal{V}_{A,\mathcal{P}}$ includes the appropriate triangle part and, at maximum, all base and connection parts multiplied by a multiplier ℓ that reflects some fractions of other triangle parts could be partly visible; thus, its surface is

$$S(\mathcal{V}_{A,\mathcal{P}}) \leq S(\mathcal{V}_{A,\mathcal{P}})_{\mathcal{B}} = S(\mathcal{T}) + \ell k \cdot S(\mathcal{B}) + \ell k \cdot S(\mathcal{C}). \tag{4}$$

Finally, for any point $A \in \mathcal{P}$ in any connection part of polygon $\mathcal{P}$, the visibility polygon $\mathcal{V}_{A,\mathcal{P}}$ includes, at maximum, all base and connection parts multiplied by a multiplier ℓ that reflects some fractions of triangle parts could be partly visible; thus, its surface is

$$S(\mathcal{V}_{A,\mathcal{P}}) \leq S(\mathcal{V}_{A,\mathcal{P}})_{\mathcal{C}} = \ell k \cdot S(\mathcal{B}) + \ell k \cdot S(\mathcal{C}). \tag{5}$$

Putting formulas (3, 4, 5) together, we get

$$\begin{aligned} S(\mathcal{V}_{A,\mathcal{P}}) &\leq \max\left\{ S(\mathcal{V}_{A,\mathcal{P}})_{\mathcal{T}}, S(\mathcal{V}_{A,\mathcal{P}})_{\mathcal{B}}, S(\mathcal{V}_{A,\mathcal{P}})_{\mathcal{C}} \right\} = \\ &= S(\mathcal{T}) + \ell k \cdot S(\mathcal{B}) + \ell k \cdot S(\mathcal{C}). \end{aligned} \tag{6}$$

Finally, since it is $a \gg \varepsilon \gtrsim 0$, it is also $S(\mathcal{T}) \propto a^2 \gg a\varepsilon = S(\mathcal{B}) = S(\mathcal{C}) \gtrsim 0$ and $1 \gg \frac{a\varepsilon}{S(\mathcal{T})} \gtrsim 0$ ($*$). Also, we may expect, that $\ell \approx k$, or, moreover, we may set $\varepsilon \gtrsim 0$ so that $\ell k \cdot \frac{a\varepsilon}{S(\mathcal{T})} \gtrsim 0$ ($\ddagger$). We get

$$\begin{aligned} \nu = \frac{S(\mathcal{V}_{A,\mathcal{P}})}{S(\mathcal{P})} &\overset{(2,6)}{\leq} \frac{S(\mathcal{T}) + \ell k \cdot S(\mathcal{B}) + \ell k \cdot S(\mathcal{C})}{k \cdot (S(\mathcal{T}) + S(\mathcal{B}) + S(\mathcal{C})) - S(\mathcal{C})} \overset{(\dagger)}{=} \\ &\overset{(\dagger)}{=} \frac{S(\mathcal{T}) + \ell k \cdot a\varepsilon + \ell k \cdot a\varepsilon}{k \cdot (S(\mathcal{T}) + a\varepsilon + a\varepsilon) - a\varepsilon} = \\ &= \frac{S(\mathcal{T}) + 2\ell k \cdot a\varepsilon}{kS(\mathcal{T}) + (2k-1) \cdot a\varepsilon} = \\ &= \frac{1 + 2\ell k \cdot \frac{a\varepsilon}{S(\mathcal{T})}}{k + (2k-1) \cdot \frac{a\varepsilon}{S(\mathcal{T})}} \overset{(*)}{=} \\ &\overset{(*)}{=} \lim_{\frac{a\varepsilon}{S(\mathcal{T})} \to 0} \frac{1 + 2\ell k \cdot \frac{a\varepsilon}{S(\mathcal{T})}}{k + (2k-1) \cdot \frac{a\varepsilon}{S(\mathcal{T})}} \overset{(\ddagger)}{=} \\ &\overset{(\ddagger)}{=} \frac{1+0}{k+0} = \frac{1}{k} \overset{(\bullet)}{=} \\ &\overset{(\bullet)}{=} \frac{1}{\lfloor n/3 \rfloor}. \end{aligned} \tag{7}$$

Thus, we showed that for any point in an n-sided saw-like polygon or on its boundary is the proportion ν of its visibility polygon's surface to an entire polygon's surface asymptotically not greater than $\frac{1}{\lfloor n/3 \rfloor}$, i.e., $\nu \lesssim \frac{1}{\lfloor n/3 \rfloor}$. □

So, for each simple n-sided polygon, there indeed exists a point in the polygon or on its boundary so that the proportion ν of its visibility polygon's surface to an entire polygon's surface is $\nu \geq \frac{1}{\lfloor n/3 \rfloor}$. However, the saw-like polygons are examples of n-sided polygons where the proportion ν is upper-bounded, so it is $\nu \lesssim \frac{1}{\lfloor n/3 \rfloor}$, and, finally, $\frac{1}{\lfloor n/3 \rfloor} \leq \nu \lesssim \frac{1}{\lfloor n/3 \rfloor}$, i.e., $\nu \approx \frac{1}{\lfloor n/3 \rfloor}$ for any point in this kind of polygon or on its boundary. So, the lower bound estimate for the surface proportion ν cannot be in general greatly improved.

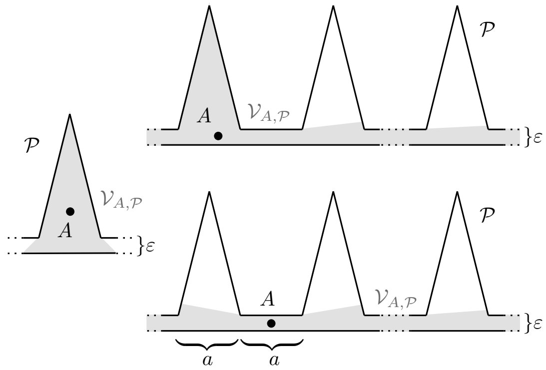

Fig. 4. Visibility polygons (in gray color) for point A in polygon $\mathcal{P}$. For any point $A \in \mathcal{P}$ the visibility polygon $\mathcal{V}_{A,\mathcal{P}}$ includes (i) the triangle part and, at maximum (!), all base and connection parts (left middle subfigure), if $A \in \mathcal{P}$ is in any triangle part of $\mathcal{P}$; (ii) the appropriate triangle part and, at maximum (!), all base and connection parts multiplied by a multiplier ℓ that reflects some fractions of other triangle parts could be partly visible, if $A \in \mathcal{P}$ is in any base part of $\mathcal{P}$ (right top subfigure); (iii) at maximum (!), all base and connection parts multiplied by a multiplier ℓ that reflects some fractions of triangle parts could be partly visible, if $A \in \mathcal{P}$ is in any connection part of $\mathcal{P}$ (right bottom subfigure).

IV. An application of the lower bound of the proportion of a visibility polygon's surface to an entire polygon's surface: garden-watering problem

Let us have a garden patch following a shape of a simple n-sided polygon, $n \geq 3$, that needs to be watered using a rotary sprinkler placed in the polygon or on its boundary. Due to the patch's loose soil and water diffusion, it is enough to water only any η-proportion of the patch's surface to hydrate the entire patch, where $0 \leq \eta \leq 1$. The sprinkler can water any point of the patch at any distance, but its water stream cannot cross the patch's boundary. Is it possible to water the patch using only one sprinkler?

Solution. Applying theorem 3, one sprinkler can surely water ν-proportion of the entire patch, where $\nu \geq \frac{1}{\lfloor n/3 \rfloor}$. Thus, if $\frac{1}{\lfloor n/3 \rfloor} \geq \eta$, one sprinkler is sufficient. Moreover, using lemma 2, very likely one sprinkler could water an even greater proportion of the patch, up to $\nu = 1$ if the patch is, e.g., convex. However, if the patch follows a shape of the saw-like polygon as introduced in definition 3 for $n \geq 6$, one sprinkler can water no more than only ν-proportion of the patch, where $\nu \lesssim \frac{1}{\lfloor n/3 \rfloor}$, as proved in theorem 4. As a footnote, the solution is non-constructive, given the garden patch follows a simple n-sided polygon's shape. So, a sprinkler of the properties mentioned above exists in the polygon or on its boundary. However, the introduced solution does not offer a way to find the exact position of the sprinkler in the polygon or on its boundary, satisfying the demanded properties. Searching for such a sprinkler position could be of high computational complexity. $\square$

V. Conclusion remarks

Having an n-sided polygon without holes and edge intersections for $n \geq 3$, there is always a point inside the polygon or on its boundary that ensures a proportion of the visibility polygon's surface for the point to the entire polygon's surface is at least $\frac{1}{\lfloor n/3 \rfloor}$, as derived using the art gallery problem. Also, there are n-sided polygons, e.g., the saw-like ones for $n \geq 6$, so that for any point in the polygon or on its boundary is the proportion of the visibility polygon's surface for such a point to the entire polygon's surface not greater than $\frac{1}{\lfloor n/3 \rfloor}$. Thus, the lower bound of the proportion of the visibility polygon's surface to the entire polygon's surface, $\frac{1}{\lfloor n/3 \rfloor}$, cannot be generally improved.

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