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An Elliptic Intuitionistic Fuzzy Portfolio Selection Problem based on Knapsack Problem

Velichka Traneva
Prof. Asen Zlatarov University
1 Prof. Yakimov Blvd, Burgas 8000, Bulgaria
Email: veleka13@gmail.com

Petar Petrov
Prof. Asen Zlatarov University
1 Prof. Yakimov Blvd, Burgas 8000, Bulgaria
Email: peshopbs2@gmail.com

Stoyan Tranev
Prof. Asen Zlatarov University
1 Prof. Yakimov Blvd, Burgas 8000, Bulgaria
Email: tranev@abv.bg

***Abstract*—This paper suggests an index-matrix approach to a knapsack-based portfolio selection model (E-IFKP) with parameters, characterized by elliptic intuitionistic fuzzy values. Elliptic Intuitionistic Fuzzy Sets are a tool to model the greater uncertainty of the environment, which is introduced in 2021. In the developed E-IFKP model, the price and the return value of the assets are determined by experts taking into account their rank. Three scenarios are proposed to the decision maker for the final choice - pessimistic, optimistic, and average. The proposed E-IFKP extends the dynamic programming approach for the Knapsack problem, which aims to select items to be placed in the knapsack to achieve the highest possible total value not exceeding its capacity. To determine the best option for an E-IFKP for certain data from the US stock exchange a software for conducting the proposed approach is developed and is used in the case study.**

I. Introduction

THE GOAL of the portfolio selection problem is to select the assets that will receive the most value from the limited resources that are available [37]. Markowitz [27], who introduced the mean-variance model and treated asset returns as random variables in the multivariate normal distribution, laid the groundwork for portfolio selection. That model defines efficient portfolios as those that maximize expected return for a given level of risk or those that minimize risk for a given level of expected return. The Markowitz portfolio selection theory and the related methods require a large amount of time sequence data. That data is required to create the statistical indices that serve as the foundation for these methods. Despite its many benefits, Markowitz's model has drawn criticism since it fails to take into account many other factors besides risk and return [50]. Over the past few decades, numerous Markowitz's model extensions have been created [58].

According to [57], historical data is either unavailable or insufficiently detailed to predict how the market will evolve in the real world. As a result, another option might be to review financial reports and the opinions of experts and/or investor preferences. Making decisions in the real financial market is frequently complicated and unclear, which is another issue. In order to take into account the actual state of the financial markets in portfolio selection models, numerous ways based on uncertain conditions have been created. These include a robust-based approach in [18], a scenario-based approach in [35], and fuzzy methods in [42], [43]. The developed theory of fuzzy logic [56] is a useful tool for working with incomplete or ambiguous information. The essential idea is to transform linguistic variables into fuzzy sets (FSs) using the appropriate membership functions [58]. It is suggested that fuzzy portfolio selection take into account the expertise of professionals, the subjective opinions of investors, or quantitative and qualitative analysis in portfolio selection challenges. Fuzzy portfolio models are another type of potential method for resolving non-probabilistic portfolio selection because the investment behaviors to new economic events cannot be precisely evaluated by the prior return rates for the selected securities due to the exclusion of many factors in the portfolio selection process. In [17], [26], [55], the objective is to maximize the fuzzy return rates while limiting the maximum investment risk by employing possibility theory, which was modeled and researched for the portfolio choice problem. Theories of possibility or believability that lead to the best selections in a fuzzy portfolio selection can be used to summarize the key studies in fuzzy portfolio models. The development of multi-objective risk measurements and fuzzy portfolio selection evaluations are presented in [30]. The entropy method is used in [55] to formulate a weighted possibility fuzzy multiobjective and higher order moment portfolio model with the efficiency and effectiveness portfolio selections. Two fuzzy-AHP approaches for portfolio selection in the Istanbul Stock Exchange are performed and compared in [43]. The capital gain tax to fuzzy portfolio selection is taken into consideration in [16] and formulated as a bi-objective mean-variance problem that is solved by a time-varying numerical integral-based particle swarm optimization algorithm. By constructing the evolutionary algorithm and fuzzy simulation approach to

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demonstrate the efficient algorithm, a skewness fuzzy variable is employed in in [24] to formulate a mean-variance-skewness fuzzy portfolio selection. In order to differentiate between three different types of risk behaviors for investors, a fuzzy portfolio selection for dealing with qualitative information that is represented as hesitant fuzzy elements is suggested in [58]. A fascinating subject in the study of fuzzy portfolio selections, according to [31], is the risk behavior analysis for investors. In [49], the proposed threshold of excess investment for each security in the portfolio selection is the assured return rate. A fuzzy analytic network technique is employed for portfolio selection [37], and a great deal of other criteria other than risk and return are studied. According to [34], there are several financial applications for fuzzy logic, including portfolio optimization. Multi-objective linear programming is created in study [54] for portfolio selection in a fuzzy environment. The model, based on the investor's risk behavior in a dimension that is different from the gap between the guaranteed return rate and the return rate for each security, is suggested in [11]. According to [21], where the mean-variance model was used for portfolio selection and the risk the behavior of an investor in a different dimension distance for shortage investment and the excess investment was still not taken into account, the adjustable security proportion for excess investment and shortage investment based on the selected guaranteed return rates for profitable returns is suggested. The dimension of excess investment has been taken into consideration as the fuzzy portfolio based on guaranteed return rates has been developed for investors with various risk preferences [10]. According to Gorzaczany [19], since decision-makers aren't always able to accurately explain an element's degree of membership, formal representations of fuzzy sets are usually insufficient. There is often a degree of hesitancy between membership and non-membership in real-world issues since decision-makers frequently express their thoughts even when they are unsure of them [53]. Fuzzy set extensions are needed to address this problem. In [1], Atanassov has proposed the intuitionistic fuzzy sets (IFSs) as an extension of FSs. The difference between FS and IFS is that the elements of the latter sets have a degree of hesitancy, which complements the corresponding sum of the element membership and non-membership degrees to 1. In [52] are described as other "extensions" of the IFSs and these extensions of IFS have been compared with themselves. The authors of [52] have demonstrated that IFS can completely describe a Hesitant Fuzzy Set [44]. In [52], the authors also prove that the Picture fuzzy sets [12], the Cubic set [25], the Neutrosophic fuzzy sets [40] and the Support-intuitionistic fuzzy sets [33] are representable by interval-valued IFSs (IVIFSs) [8]. In recent years, two more generalizations of intuitionistic fuzzy sets have appeared in the form of circular [4], and elliptic IFSs [5], which also generalize interval-valued IFSs. The ultimate goal of an intuitionistic fuzzy interactive multi-objective optimization approach is to find the optimum solution that maximizes satisfaction and minimizes unhappiness, according to [36]. Interactive optimization techniques' primary objective is to actively involve the decision-maker in problem-solving. A socially responsible portfolio selection problem is solved in [20] utilizing an interactive triangular intuitionistic fuzzy approach. A portfolio selection model built on the knapsack problem with interval uncertainty is provided in the study [51]. It is suggested that the created model, which is based on the knapsack problem (KP), can be used to appropriately allocate the number of shares to various assets and may be able to determine the best asset allocation under unique circumstances involving relatively high stock values.

In this regard, our efforts are to develop an extension of the portfolio selection problem so that it can be applied to IF data and then to circular and elliptic IFSs. In our previous works [45], [46], we for the first time have suggested IF and circular IF KP (C-IFKP) for finding the optimal solution respectively of the IF and circular IF portfolio selection problem. The main parameters in the problems are IF pairs or circular IF triples, determined by experts under three different scenarios - pessimistic, optimistic, and average. E-IFSs are described as sets with an ellipse indicating the degrees of membership and non-membership for each element of the universe [5]. No developed models for optimal portfolio selection with elliptic IF data were found in the Scopus database. The index-matrix method to an elliptic knapsack-based portfolio selection model (E-IFKP), which extends the Circular IFKP and IFKP approach from the studies [46], is introduced here. Experts agree on the importance, cost, and return of each asset, and the suggested approach takes these ratings into account. Pessimistic, optimistic, and average scenarios are put out to the decision-maker for consideration before making a final decision. Software for carrying out the suggested E-IFKP is under development and utilized in the real case at a specific time to find the best alternative for an E-IFKP for specific data at a specific moment from the US stock exchange. The advantage of this model is that it can be applied to both plain and elliptic IF data. Another advantage is that it can easily be extended so that it can be applied to multidimensional IF data. Theoretical Contributions of the study are: the introduced definition of elliptic IF quads; extended comparison operations and relations on IF pairs to those on elliptic IF quads.

The Knapsack problem's (KP) goal is to maximize the total utility value of all things selected by the decision-maker within the constraints of a knapsack [28]. The phrase "Knapsack problem" first appeared in early publications by George Dantzig in the 1950s [13]. Gilmore and Gomory examined the dynamic programming method to the KP in 1966 [15]. An approximation approach for the solution of a multiple choice fuzzy KP (FKP) is provided in [22]. Ant colony optimization with environmental changes is developed in [32]. The paper presents one kind of FKP [38]. A dynamic programming strategy has been provided in [9], [38], [39] for solving FKP. In the work [14], an approach for ant colonies to optimize the Multiple-Constraint Knapsack Problem utilizing intuitionistic fuzzy (IF) pheromone updating is described. The idea of the E-IFKP and its usage for the portfolio selection problem according to three scenarios give this work its novelty.

The remaining portions of this study are structured as follows: Short remarks to the elliptic intuitionistic fuzzy quads (E-IFQs) and the index matrices (IMs) are provided in Section II. A form of 0-1 E-IFKP for portfolio selection is suggested in Section III, and with the aid of software, it is applied to a real E-IFKP for the selection of portfolio shares of the IT companies which make up the Dow Jones Industrial Average. Section IV, which summarizes the findings and offers recommendations for additional study, brings the work to a close.

II. Remarks on Elliptic Intuitionistic Fuzzy Quads and Index Matrices

We will define elliptic intuitionistic fuzzy quads (E-IFQs), index matrices (IMs), as well as some of their operations and relationships, in this section. In 2021, the IFS is extended with the E-IFS, which has a different interface. An ellipse with semi-major and semi-minor axes exists around each element of E-IFS that represents its membership degree and non-membership degree [5].

Let's consider the definition of an intuitionistic fuzzy pair (IFP) [7]: an IFP has the form of $\langle a(p), b(p)\rangle$ or $\langle \mu(p), \nu(p)\rangle$: The components of an IFP are $a(p)(\mu(p)), b(p)(\nu(p)) \in [0,1]$ and $a(p)+b(p) = \mu(p)+\nu(p) \leq 1$, respectively. These elements represent the degrees of membership and non-membership of a proposition p. Using the definition of the E-IFS [5], let us define E-IFQ as an object of the following form:

$$\langle a(p), b(p); u, v\rangle = \langle \mu(p), \nu(p); u, v\rangle,$$

where $a(p)+b(p) = \mu(p)+\nu(p) \leq 1$, which is utilized to evaluate the statement p, is regarded as the "truth degree" and "falsity degree" of the assertion p, respectively. The semi-major and semi-minor axes of the ellipse with the center $\langle a(p)(\mu(p)), b(p)(\nu(p))\rangle$ are $u, v \in [0, \sqrt{2}]$, respectively.

Two E-IFQs $x_{u_1,v_1} = \langle a, b; u_1, v_1\rangle$ and $y_{u_2,v_2} = \langle c, d; u_2, v_2\rangle$, shall be used. Let us define an operation $* \in \{\min, \max\}$. The operations over E-IFQs that follow are based on the E-IFSs operations from [5]. For the E-IFQs, the operations "subtraction" and "division" for C-IFPs [46] are modified.

$$\neg x_{u_1,v_1} = \langle b, a; u_1, u_2\rangle;$$
$$x_{u_1,v_1} \wedge_* y_{u_2,v_2} = \langle \min(a,c), \max(b,d); *(u_1,u_2), *(v_1,v_2)\rangle;$$
$$x_{u_1,v_1} \vee_* y_{u_2,v_2} = \langle \max(a,c), \min(b,d); *(u_1,u_2), *(v_1,v_2)\rangle;$$
$$x_{u_1,v_1} +_* y_{u_2,v_2} = \langle a+c-a.c, b.d; *(u_1,u_2), *(v_1,v_2)\rangle;$$
$$x_{u_1,v_1} \bullet_* y_{u_2,v_2} = \langle a.c, b+d-b.d; *(u_1,u_2), *(v_1,v_2)\rangle;$$
$$x_{u_1,v_1} @_* y_{u_2,v_2} = \langle \tfrac{a+c}{2}, \tfrac{b+d}{2}; *(u_1,u_2), *(v_1,v_2)\rangle$$
$$x_{u_1,v_1} -_* y_{u_2,v_2} = \langle \max(0, a-c), \min(1, b+d, 1-a+c); *(u_1,u_2), *(v_1,v_2)\rangle$$
$$x_{u_1,v_1} :_* y_{u_2,v_2} = \begin{cases} \langle \min(1, a/c), \min(\max(0, 1-a/c), \\ \max(0, (b-d)/(1-d))); *(u_1,u_2), *(v_1,v_2)\rangle \\ \text{if } c \neq 0 \ \&d \ \neq 1 \\ \langle 0, 1; *(u_1,u_2), *(v_1,v_2)\rangle \text{ otherwise} \end{cases}$$

Since the semi-major and semi-minor axes produce outputs with minimal and maximum degrees of uncertainty, respectively, the operations presented here are based on their minimum and maximum values. We propose the following relation for comparing E-IFs using a formula for the distance between C-IFSs [6], the relation for comparing two C-IFPs [46], and the distance from the element to the ideal positive alternative [41].

$$x_{u_1,v_1} \geq_{R^{elliptic}} y_{u_2,v_2} \quad \text{iff } R^{elliptic}_{x_{u_1,v_1}} \leq R^{elliptic}_{y_{u_2,v_2}} \tag{1}$$

where

$$R^{elliptic}_{x_{u_1,v_1}} = \frac{1}{6}(2-a-b)\big(|\sqrt{2}-u_1| + |\sqrt{2}-v_1| + |1-a|\big)$$

is the distance between x and the ideal positive alternative $\langle 1, 0; \sqrt{2}, \sqrt{2}\rangle$ to x. According to the Szmidt and Kacprzyk's version of the distance [6], we state that the E-IFQs x_{u_1,v_1} and y_{u_2,v_2} are in α-proximity ($\alpha \in [0;1]$): if

$$d(x_{u_1,v_1}, y_{u_2,v_2})$$
$$= \frac{1}{3}\big(|u_1-u_2| + |v_1-v_2| + 0.5(|a-c| + |b-d| + |c+d-a-b|)\big) \leq \alpha$$

In 1987, according to [2], the theory of index matrices (IMs) was developed. Over IMs, several operations, relations, and operators are defined (see [3], [48]). Assume that the set of indices $\mathscr{I}$ is fixed. Two-dimensional elliptic intuitionistic fuzzy index matrix (2-D E-IFIM) $A = [K, L, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j}; u_{k_i,l_j}, v_{k_i,l_j}\rangle\}]$ with index sets K and L ($K, L \subset \mathscr{I}$), we denote the object analogous to circular IFIM (C-IFIM) [46]:

	l_1	$\ldots$	l_n
k_1	$\langle \mu_{k_1,l_1}, \nu_{k_1,l_1}; u_{k_1,l_1}, v_{k_1,l_1}\rangle$	$\ldots$	$\langle \mu_{k_1,l_n}, \nu_{k_1,l_n}; u_{k_1,l_n}, v_{k_1,l_n}\rangle$
$\vdots$	$\vdots$	$\ddots$	$\vdots$
k_m	$\langle \mu_{k_m,l_1}, \nu_{k_m,l_1}; u_{k_m,l_1}, v_{k_m,l_1}\rangle$	$\ldots$	$\langle \mu_{k_m,l_n}, \nu_{k_m,l_n}; u_{k_m,l_n}, v_{k_m,l_n}\rangle$

The definition of a 3-D E-IFIM extends the 2-D E-IFIM concept and is identical to those of the 3-D IM, presented in the [3]. Let us introduce some operations over the E-IFIMs.

Transposition [3]: The transposed IM of A is A'.

Let us introduce the following operations over E-IFIMs $A = [K, L, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j}; rf_{k_i,l_j}, rs_{k_i,l_j}\rangle\}]$ and $B = [P, Q, \{\langle \rho_{p_r,q_s}, \sigma_{p_r,q_s}\rangle; \delta f_{k_i,l_j}, \delta s_{k_i,l_j}\}]$ with a similar form to that of [3], [46].

Addition-($\circ_1, \circ_2, *$):
$A \oplus_{(\circ_1,\circ_2,*)} B = [K \cup P, L \cup Q, \{\langle \phi_{t_u,v_w}, \psi_{t_u,v_w}\rangle; \eta f_{t_u,v_w}, \eta s_{t_u,v_w}\}]$, where $\langle \circ_1, \circ_2\rangle \in \{\langle \max, \min\rangle, \langle \min, \max\rangle, \langle$ average, average$\rangle\}$ and $* \in \{\max, \min\}$.
$\langle \phi_{t_u,v_w}, \psi_{t_u,v_w}; \eta f_{t_u,v_w}, \eta s_{t_u,v_w}\rangle = \langle \circ_1(\mu_{k_i,l_j}, \rho_{p_r,q_s}), \circ_2(\nu_{k_i,l_j}, \sigma_{p_r,q_s}); *(rf_{t_u,v_w}, \delta f_{t_u,v_w})), *(rs_{t_u,v_w}, \delta s_{t_u,v_w})\rangle$.

Termwise subtraction-(max,min):

$$A -_{(\max,\min,*)} B = A \oplus_{(\max,\min,*)} \neg B.$$

Termwise multiplication:
$A \otimes_{(\circ_1,\circ_2,*)} B = [K \cap P, L \cap Q, \{\langle \phi_{t_u,v_w}, \psi_{t_u,v_w}; \eta f_{t_u,v_w}, \eta s_{t_u,v_w}\rangle\}]$, where $\langle \phi_{t_u,v_w}, \psi_{t_u,v_w}; \eta f_{t_u,v_w}, \eta s_{t_u,v_w}\rangle = \langle \circ_1(\mu_{k_i,l_j}, \rho_{p_r,q_s}), \circ_2(\nu_{k_i,l_j}, \sigma_{p_r,q_s}); *(rf_{t_u,v_w}, \delta f_{t_u,v_w}, *(rs_{t_u,v_w}, \delta s_{t_u,v_w})\rangle$.

The following operations have no equivalents with these verso classical matrices. They are developed to be able to automate certain actions on IMs in order to implement various models and algorithms.

Reduction [3]: An IM A's operations-reduction $(k, \perp)$ is defined as follows:

$A_{(k,\perp)} = [K-\{k\}, L, \{c_{t_u,v_w}\}]$, where $c_{t_u,v_w} = a_{k_i,l_j} (t_u = k_i \in K-\{k\}, v_w = l_j \in L)$.

Projection [3]: Let $M \subseteq K$ and $N \subseteq L$. Then, $pr_{M,N}A = [M,N,\{b_{k_i,l_j}\}]$, where for each $k_i \in M$ and each $l_j \in N$, $b_{k_i,l_j} = a_{k_i,l_j}$.

Substitution [3]: $\left[\frac{p}{k};\perp\right]A = \left[(K-\{k\})\cup\{p\}, L, \{a_{k,l}\}\right]$

Internal subtraction of IMs' components [48]:
$IO_{-_{(\max,\min)}}(\langle k_i,l_j,A\rangle, \langle p_r,q_s,B\rangle) = [K,L,\{\langle \gamma_{t_u,v_w}, \delta_{t_u,v_w}\rangle\}]$.

Index type operations [48]:
$AGIndex_{(\max_R^{elliptic}),(\not\perp)}(A) = \langle k_i,l_j\rangle$, where $\langle k_i,l_j\rangle$ (for $1 \le i \le m, 1 \le j \le n$) is the index of the maximum E-IFQ of A in the sense of the relation (1) that has no empty value.
$Index_{(\max_R^{elliptic}),k_i}(A) = \{\langle k_i,l_{v_1}\rangle,\ldots,\langle k_i,l_{v_x}\rangle,\ldots,\langle k_i,l_{v_V}\rangle\}$, where $\langle k_i,l_{v_x}\rangle$ (for $1 \le i \le m, 1 \le x \le V$) are the indices of the largest element in A's k_i-th row.

Aggregation operations Let us extend the operations $\#_q, (q \le i \le 3)$ from [47] such that they can be applied over E-IFQs $x = \langle a,b;rf_1,rs_1\rangle$ and $y = \langle c,d;rf_2,rs_2\rangle$:
$x\#_1,*y = \langle min(a,c), max(b,d); *(rf_1,rf_2), *(rs_1,rs_2)\rangle$;
$x\#_2,*y = \langle average(a,c), average(b,d); *(rf_1,rf_2), *(rs_1,rs_2)\rangle$;
$x\#_3,*y = \langle max(a,c), min(b,d); *(rf_1,rf_2), *(rs_1,rs_2)\rangle$.

Let the fixed index be $k_0 \notin K$. The expanded definition of the aggregation operation $\alpha_{K,\#_q,*}(A,k_0)$ by the dimension K over 3-D E-IFIM A utilizing that of ([46], [47]) is as follows:

$h_g \in H$	l_1	$\ldots$
k_0	$\underset{i=1}{\overset{m}{\#_q,*}} \langle \mu_{k_i,l_1,h_g}, \nu_{k_i,l_1,h_g}; rf_{k_i,l_1,h_g}, rf_{k_i,l_1,h_g}\rangle$	$\ldots$

$\ldots$	l_n
$\ldots$	$\underset{i=1}{\overset{m}{\#_q,*}} \langle \mu_{k_i,l_n,h_g}, \nu_{k_i,l_n,h_g}; rf_{k_i,l_1,h_g}, rf_{k_i,l_1,h_g}\rangle$

,

Aggregate global internal operation [48]: $AGIO_{\oplus_{(\#_q,*)}}(A)$. If $q=1, q=2$ or $q=3$, we get a pessimistic, averaged, or optimistic scenario.

Operation "Purge" of IM A The following is how we define the new operation "Purge" by the dimension K as follows: $Purge_K(A)$ decreases each row k_x of A, if $a_{k_x,l_j} \le a_{k_y,l_j}$, but $a_{k_x,l_e} \ge a_{k_y,l_e}$ for $1 \le x \le m$, $1 \le y \le m$, $1 \le j \le n$ and $1 \le e \le n$.

III. An Elliptic Intuitionistic Fuzzy Portfolio Selection Problem based on Knapsack Problem

In this part, we extend the dynamic programming strategy for C-IFKP [46] to develop an IM interpretation for a set method for 0-1 E-IFKP for the portfolio selection problem. The problem is: An investor has a budget of $Bu = \langle \rho, \sigma; rf_{Bu}, rs_{Bu}\rangle$ to spend on assets. A set of m assets from $\{k_1,\ldots,k_i,\ldots,k_m\}$ must be evaluated by experts $\{d_1,\ldots,d_s,\ldots,d_D\}$ (for $s = 1,...,D$) with a given IFP rating $re_s = \langle \delta_s, \varepsilon_s\rangle$ $(1 \le s \le D)$ using the criteria c_1 and c_2. Let us use the symbols a_{k_i,c_1} (for $i = 1,...,m$) and a_{k_i,c_2} (for $i = 1,...,m$) to represent the return and the price, respectively, of the k_i-th asset. The objective of the problem is to choose the investor's portfolio's assets as optimally as possible while staying within his financial constraints. The parameters of this optimization problem are highly unknown because of market dynamics. Through the use of the E-IF logic toolkit, helped by IMs, it is required to look for the best answer for the investment portfolio with three decision-making scenarios (optimistic, averaged, and pessimistic).

A. An Elliptic IF Portfolio Selection Problem with a Dynamic Programming Approach Using a Type E-IFKP

The following are the stages in the IM interpretation for the Elliptic Intuitionistic Fuzzy Portfolio Selection Problem, based on the Circular IFKP technique [46]:

Step 1. 3-D evaluation IFIM $EV[K,C,E,\{ev_{k_i,c_j,d_s}\}]$ is created in compliance with the aforementioned problem, where $K = \{k_1,k_2,\ldots,k_m\}$, $C = \{c_1,c_2\}$, $E = \{d_1,\ldots,d_s,\ldots,d_D\}$ and the element $\{ev_{k_i,c_j,d_s}\} = \langle \mu_{k_i,c_j,d_s}, \nu_{k_i,c_j,d_s}\rangle$ (for $1 \le i \le m, 1 \le j \le n, 1 \le s \le D$) is the estimate of the d_s-th expert for the k_i-th asset by the c_j-th criterion $(j = 1,2)$. Due to changes in some uncontrolled elements, the expert is unsure about the evaluation according to a particular criterion, and his evaluations take the shape of IFPs. Let the s-th expert's score (rating) $re_s, s \in E$ be specified by an IFP $\langle \delta_s, \varepsilon_s\rangle$, where δ_s and ε_s are considered as the expert's level of competence and incompetence, respectively.

Next, we calculate

$$EV^*[K,C,E,\{ev^*_{k_i,c_g,d_s}\}] = re_1 pr_{K,C,d_1} EV \oplus_{(\circ_1,\circ_2)} \cdots$$
$$\cdots \oplus_{(\circ_1,\circ_2)} re_D pr_{K,C,d_D} EV.$$
$$EV := EV^*(ev_{k_i,l_j,d_s} = ev^*_{k_i,l_j,d_s},\ \forall k_i \in K, \forall l_j \in L, \forall d_s \in E).$$

The degrees of membership and non-membership of the E-IFQs are determined using the three aggregating operations $\alpha_{K,\#_1,*}, \alpha_{K,\#_3,*}$ and $\alpha_{K,\#_2,*}$, which provide the evaluations of the k_i-th asset against the c_j-th criterion in a present moment $h_f \notin E$:

$PI_{min}[K,C,h_m,\{pi_{min_{k_i,c_g,h_f}}\}] = \alpha_{E,\#_1}(EV^*,h_m)$

h_m	c_1	c_2
k_1	$\underset{s=1}{\overset{D}{\#_1}} \langle \mu_{k_1,c_1,d_s}, \nu_{k_1,c_1,d_s}\rangle$	$\underset{s=1}{\overset{D}{\#_1}} \langle \mu_{k_1,c_2,d_s}, \nu_{k_1,c_2,d_s}\rangle$
$\vdots$	$\vdots$	$\vdots$
k_m	$\underset{s=1}{\overset{D}{\#_1}} \langle \mu_{k_m,c_1,d_s}, \nu_{k_m,c_1,d_s}\rangle$	$\underset{s=1}{\overset{D}{\#_1}} \langle \mu_{k_m,c_2,d_s}, \nu_{k_m,c_2,d_s}\rangle$

$PI_{max}[K,C,h_m,\{pi_{max_{k_i,c_g,h_f}}\}] = \alpha_{E,\#_3}(EV^*,h_m) =$

h_m	c_1	c_2
k_1	$\underset{s=1}{\overset{D}{\#_3}} \langle \mu_{k_1,c_1,d_s}, \nu_{k_1,c_1,d_s}\rangle$	$\underset{s=1}{\overset{D}{\#_3}} \langle \mu_{k_1,c_2,d_s}, \nu_{k_1,c_2,d_s}\rangle$
$\vdots$	$\vdots$	$\vdots$
k_m	$\underset{s=1}{\overset{D}{\#_3}} \langle \mu_{k_m,c_1,d_s}, \nu_{k_m,c_1,d_s}\rangle$	$\underset{s=1}{\overset{D}{\#_3}} \langle \mu_{k_m,c_2,d_s}, \nu_{k_m,c_2,d_s}\rangle$

Then construct $PI* = PI_{min} \oplus_{(\circ_1,\circ_2,*)} PI_{max}$ and $PI[K,C,h_f,\{pi_{k_i,c_g,h_f}\}] = \alpha_{E,\#_2}(PI^*,h_f)$, whose elements are the coordinates of the centers of the E-IFQs evaluating the shares.

We now determine E-IFIM $A[K,C\{a_{k_i,c_g}\}]$, which represents current evaluations of the assets utilizing the approach from [5] by criteria for return and price:

	c_1	c_2
k_1	$\langle \mu^a_{k_1,c_1}, \nu^a_{k_1,c_1}; rf^a_{k_1,c_1}, rs^a_{k_1,c_1}\rangle$	$\langle \mu^a_{k_1,c_2}, \nu^a_{k_1,c_2}; rf^a_{k_1,c_2}, rs^a_{k_1,c_2}\rangle$
$\vdots$	$\vdots$	$\vdots$
k_m	$\langle \mu^a_{k_m,c_1}, \nu^a_{k_m,c_1}; rf^a_{k_m,c_1}, rs^a_{k_m,c_1}\rangle$	$\langle \mu^a_{k_m,c_2}, \nu^a_{k_m,c_2}; rf^a_{k_m,c_2}, rs^a_{k_m,c_2}\rangle$

where $K = \{k_1, \ldots, k_i, \ldots, k_m\}, i = 1, \ldots, m; C = \{c_1, c_2\}$, a_{k_i,c_g} (for $i = 1, \ldots, m; g = 1,2$) are created as E-IFQs by converting the IFPs pi_{k_i,c_j,d_s} using the following steps

for $g = 1$ to 2, $i = 1$ to m $\Big\{\mu^a_{k_i,c_g} = \mu^{pi}_{k_i,c_g,h_f}; \nu^a_{k_i,c_g} = \nu^{pi}_{k_i,c_g,h_f},$

$rf^a_{k_i,c_g}$

$$= \sqrt{\left(\min_{1\le s\le D}\mu^{ev}_{k_i,c_g,d_s}{}^2 + \left\{\frac{\max_{1\le s\le D}\mu^{ev}_{k_i,c_g,d_s}{}^2 - \min_{1\le s\le D}\mu^{ev}_{k_i,c_g,d_s}{}^2}{\max_{1\le s\le D}\nu^{ev}_{k_i,c_g,d_s}{}^2 - \min_{1\le s\le D}\nu^{ev}_{k_i,c_g,d_s}{}^2}\right\}\cdot \min_{1\le s\le D}\nu^{ev}_{k_i,c_g,d_s}{}^2\right)^2}$$

and $rs^a_{k_i,c_g}$

$$= \sqrt{\left(\min_{1\le s\le D}\mu^{ev}_{k_i,c_g,d_s}{}^2 \cdot \left\{\frac{\max_{1\le s\le D}\nu^{ev}_{k_i,c_g,d_s}{}^2 - \min_{1\le s\le D}\nu^{ev}_{k_i,c_g,d_s}{}^2}{\max_{1\le s\le D}\mu^{ev}_{k_i,c_g,d_s}{}^2 - \min_{1\le s\le D}\mu^{ev}_{k_i,c_g,d_s}{}^2}\right\} + \min_{1\le s\le D}\nu^{ev}_{k_i,c_g,d_s}{}^2\right)^2}$$

$\Big\}$

The input data for the portfolio's budget is then checked to ensure that it does not exceed the investor's specified budget, Bu. If the price of a given asset k_i exceeds the budget Bu, then the corresponding row of IM A is reduced by it.

for $i = 1$ to m {If $a_{k_i,c_2} > Bu$ then $A_{(k_i,\perp)}$ }

Let us denote by $|K| = m$ the number of the elements of the set K, then $|C| = 2$. As well, we define E-IFIM $X[K,C]$ containing the elements x_{k_i,c_g} (for $1 \le i \le m, \ \le g \le 2$) and:

$$\{x_{k_i,c_g}\} \in \begin{cases} \langle 1,0;0,0\rangle, & \text{if the request } k_i \text{ is selected} \\ \langle 0,1;0,0\rangle & \text{otherwise} \end{cases}$$

Let us assume that at the start of the algorithm, all components of IM X are identical to $\langle 0,1;0,0\rangle$.

Create IM

$$S^0[u_0, L] = \begin{array}{c|cc} & c_1 & c_2 \\ \hline u_0 & s^0_{u_0,c_1} & s^0_{u_o,c_2}\end{array} = \begin{array}{c|cc} & c_1 & c_2 \\ \hline u_0 & \langle 0,1;0,0\rangle & \langle 0,1;0,0\rangle\end{array}$$

Step 2. for $i = 1$ to m do $\{$Create IMs

$$R_i[k_i, C] = pr_{k_i,C}A; SH^{i-1}_1 = \left[\frac{u_i}{u_{i-1}}; \perp\right] S^{i-1}$$

for $h = 1$ to $i+1$ do {

$$SH^{i-1}_1 = SH^{i-1}_1 \oplus_{(\circ_1,\circ_2,*)} \left[\frac{u_h}{k_i}; \perp\right] R_i$$

}

$$S^i[U^i, L] = S^{i-1} \oplus_{((\circ_1,\circ_2,*))} SH^{i-1}_1;$$

for $h = 1$ to $i+1$ do $\{$ *Checks the conditions for the capacity of the knapsack*

If $s^i_{h,w} > Bu$ then $S^i_{(h,\perp)}$ }

The "Purge" procedure is currently underway by $S^i = Purge_{U^i} S^i$ } Go to *Step 3.*

Step 3. This step finds the index of the highest stock return by

$$Index_{(\max_{R elliptic}),c_1}(A) = \langle u_g, c_1\rangle$$

for $i = m$ to 1 do

{Find the α-nearest elements of $s^i_{u_g,c_1}$ (or $s^i_{u_g,c_2}$) ($\alpha = 0.5$) of S^i and choose the closest element from them - $s^i_{u_{g*},c_1}$ (or $s^i_{u_{g*},c_2}$).

If $\{s^i_{u_g*,c_1}$ (or $s^i_{u_{g*},c_2})\} \in S^i$ and $\{s^i_{u_g*,c_1}$ (or $s^i_{u_{g*},c_2})\} \notin S^{i-1}$ then

$\{x_{k_i,p} = \langle 1,0;0,0\rangle$ and $x_{k_i,w} = \langle 1,0;0,0\rangle;$

$s^i_{u_g,c_1} = s^i_{u_g*,c_1} -_* a_{k_i,c_1}; s^i_{u_g,c_2} = s^i_{u_g*,c_2} -_* a_{k_i,c_2}\}$

Go to *Step 4.*

Step 4. The optimal return and price of the investment portfolio are:

$$AGIO_{\oplus_{(\#_q,*)}}\left(pr_{K,c_1}A \otimes_{(\circ_1,\circ_2,*)} pr_{K,c_1}X\right);$$
$$AGIO_{\oplus_{(\#_q,*)}}\left(pr_{K,c_2}A \otimes_{(\circ_1,\circ_2,*)} pr_{K,c_2}X\right).$$

If $q = 1, q = 2$ or $q = 3$ then we determine the optimal benefit's pessimistic, averaging, or optimistic value. The optimistic scenario has been accepted if $\langle\circ_1,\circ_2\rangle = \langle\max,\min\rangle$ is used in all operations of the algorithm. On the other hand, if $\langle circ_1, circ_2\rangle = \langle\min,\max\rangle$ is employed, the pessimistic scenario is used. The operation "$* = \max$" is used when there is more ambiguity, else "$* = \min$". Therefore several optimal solutions could be generated according to the investor's opinion. Thus, an investor may have greater confidence in the obtained solution.

The complexity of the normal dynamic programming algorithm [28], [59] and the E-IFKP approach are both comparable - $O(m.C)$.

To study how the algorithm affects a range of input data, we are now developing software that uses the E-IFKP approach. After reading a file with a matrix of the return predictions and stock price, it completes the aforementioned procedures. It was developed in C++. This objective was accomplished by building an IM structure using the STL's std::tuple types. This structure is then used to create the fundamental IM protocols [29]. The app requires the share E-IFQs and the knapsack budget Bu as input. When the program is finished, a suggested solution is displayed on the computer screen along with a thorough output of each algorithm iteration.

In the scientific literature, no portfolio optimization model on elliptic IF fuzzy data was found, a suitable tool for representing vague or incomplete data in conditions of large fluctuations in market parameters. In this model, there are three scenarios according to the attitudes of the decision maker. Evaluations of the returns and the price of financial assets for the purpose of optimal selection of the portfolio are carried out by experts and their rating is taken into account in the evaluation process. Therefore, the developed E-IF optimal portfolio selection task is socially oriented and reflects the preferences of both the experts and the decision maker. The Markowitz [27] portfolio cannot be applied under conditions of fuzziness and large parameter fluctuations and his model cannot reflect the investor's attitude towards the market environment – whether it is pessimistic, optimistic or average.

B. An E-IFKP Real Case Study for Portfolio Selection

Here, a real case study for the best asset selection for the investor's portfolio within his budget $B = \langle 0.99, 0.0; \sqrt{2}, \sqrt{2} \rangle$ clarifies the proposed E-IFKP in this part. A group of the experts $\{d_1, d_2, d_3\}$ with the specified IFP rating $re_s = \langle \delta_s, \varepsilon_s \rangle$ $(1 \le s \le D)$ is required to evaluate a set of assets $\{k_1, k_2, k_3, k_4\}$ from the IT firms Microsoft Corp., Apple Inc, Cisco Systems Inc., and Intel Corporation, which make up the Dow Jones Industrial Average for the last 5 years, by the criteria c_1 and c_2: the return a_{k_i,c_1} (for $i = 1, ..., m$) of the k_i-th asset and its price as a_{k_i,c_2} (for $i = 1, ..., m$). The objective of the problem is to choose the investor's portfolio's assets as efficiently as possible while staying within his financial constraints using three different decision-making scenarios.

The solution to the problem:

Step 1. The initial form of the 3-D evaluation IFIM $EV[K, C, E, \{ev_{k_i,c_g,d_s}\}]$ is the following:

$$EV = \left\{ \begin{array}{c|cc} d_1 & c_1 & c_2 \\ \hline k_1 & \langle 0.54, 0.29 \rangle & \langle 0.35, 0.48 \rangle \\ k_2 & \langle 0.552, 0.31 \rangle & \langle 0.38, 0.514 \rangle \\ k_3 & \langle 0.546, 0.265 \rangle & \langle 0.399, 0.461 \rangle \\ k_4 & \langle 0.486, 0.316 \rangle & \langle 0.144, 0.694 \rangle \end{array} \right. ,$$

$$\begin{array}{c|cc} d_2 & c_1 & c_2 \\ \hline k_1 & \langle 0.504, 0.365 \rangle & \langle 0.2, 0.752 \rangle \\ k_2 & \langle 0.504, 0.32 \rangle & \langle 0.238, 0.74 \rangle \\ k_3 & \langle 0.675, 0.154 \rangle & \langle 0.0099, 0.865 \rangle \\ k_4 & \langle 0.672, 0.157 \rangle & \langle 0.024, 0.96 \rangle \end{array} ,$$

$$\left. \begin{array}{c|cc} k_1 & \langle 0.65, 0.13 \rangle & \langle 0.035, 0.944 \rangle \\ k_2 & \langle 0.081, 0.91 \rangle & \langle 0.0019, 0.982 \rangle \\ k_3 & \langle 0.072, 0.91 \rangle & \langle 0.0024, 0.99 \rangle \\ k_4 & \langle 0.126, 0.823 \rangle & \langle 0.007, 0.98 \rangle \end{array} \right\} ,$$

$\{ev_{k_i,c_j,d_s}\}$ (for $1 \le i \le 4, 1 \le g \le 2, 1 \le s \le 3$) is the expert's assessment in accordance with the c_g-th criterion for the k_i-th stock. Assign the experts the corresponding rating coefficients shown below:

$$\{r_1, r_2, r_3\} = \{\langle 0.9, 0.1 \rangle, \langle 0.8, 0.08 \rangle, \langle 0.7, 0.07 \rangle\}.$$

We create

$$EV^*[K, C, E, \{ev^*{}_{k_i,c_g,d_s}\}]$$
$$= re_1 pr_{K,C,d_1} EV \oplus_{(\circ_1,\circ_2)} \ldots \oplus_{(\circ_1,\circ_2)} re_D pr_{K,C,d_D} EV;$$
$$EV := EV^*.$$

Then, we determine E-IFIM $A[K, C]$, which consists of the assessments of the shares made at a moment h_f by 2 criteria:

$$\begin{array}{c|cc} & c_1 & c_2 \\ \hline k_1 & \langle 0.55, 0.29; 0.56, 0.98 \rangle & \langle 0.38, 0.49; 0.47, 0.68 \rangle \\ k_2 & \langle 0.5, 0.34; 0.51, 0.97 \rangle & \langle 0.19, 0.72; 0.33, 0.77 \rangle \\ k_3 & \langle 0.66, 0.14; 0.8, 0.22 \rangle & \langle 0.02, 0.91; 0.011, 1.76 \rangle \\ k_4 & \langle 0.099, 0.87; 0.09, 1.3 \rangle & \langle 0.004, 0.99; 0.003, 1.21 \rangle \end{array} ,$$

where $K = \{k_1, k_2, k_3, k_4\}$, $C = \{c_1, c_2\}$ and $\{a_{k_i,c_1}, a_{k_i,c_2}\}$ are, respectively, the k_i-th stock's price and E-IF return. $X[K, C]$ is formed with the elements $\langle 0, 1; 0, 0 \rangle$.

Step 2. The following IMs are calculated consecutively by the algorithm: $S_1^0[u_1, C], S^1[U_1, C], S_1^1[U*_1, C], S^2[U_2, C]$. The operation "Purge" has reduced the u_2 and u_3 row of $S^2[U_2, C]$. Then IMs are created: $S_1^2[U*_2, C], S^3[U_3, C]$. The operation "Purge" has reduced the u_4 row of $S^3[U_3, C]$. Then IMs are created: $S_1^3[U*_3, C], S^4[U_4, C]$. The following is the final IM $S^4[U*_4, C]$ after the "Purge" operation:

$$\begin{array}{c|cc} & c_1 & c_2 \\ \hline u_1 & \langle 0.546, 0.285; 0.56, 0.99 \rangle & \langle 0.375, 0.487; 0.47, 0.69 \rangle \\ u_2 & \langle 0.495, 0.341; 0.51, 0.97 \rangle & \langle 0.191, 0.723; 0.33, 0.69 \rangle \\ u_3 & \langle 0.546, 0.285; 0.56, 0.22 \rangle & \langle 0.022, 0.913; 0.01, 0.69 \rangle \\ u_4 & \langle 0.099, 0.867; 0.09, 0.99 \rangle & \langle 0.004, 0.986; 0.003, 0.69 \rangle \end{array}$$

Step 3. In this step, using the results from Step 2. we determine that the fourth, second, and first IT businesses' stocks are included in the investor's ideal portfolio in this problem.

Step 4. The outcomes for the optimistic, pessimistic, and average scenarios for the greatest benefit are shown in the following table (cf. table I):

TABLE I
THE RESULTS FOR THE OPTIMISTIC, PESSIMISTIC, AND AVERAGE SCENARIOS.

Scenario	c_1	c_2
Optimistic	$\langle 0.099, 0.867; 0.56, 1.31 \rangle$	$\langle 0.19, 0.72; 0.47, 0.77 \rangle$
Pessimistic	$\langle 0.099, 0.867; 0.093, 0.966 \rangle$	$\langle 0.19, 0.72; 0.33, 0.685 \rangle$
Average	$\langle 0.38, 0.498; 0.389, 1.085 \rangle$	$\langle 0.19, 0.73; 0.269, 0.89 \rangle$

In conditions of high inflation and great uncertainty, the decision-maker will choose the pessimistic scenario, in case of small fluctuations in the market parameters, the decision-maker will prefer the averaged scenario, and in the case of stability of the market parameters, the optimistic scenario will be preferred.

A comparative analysis between the proposed E-IFKP method for portfolio optimization could not be performed because we could not find methods for similar type of problems under conditions of high uncertainty modeled by E-IF logic.

After the results are obtained in E-IFKP portfolio method, the question arises whether small deviations in the values of the input parameters used change the results of the model. Checking the robustness of the results in the developed model and analyzing the sensitivity to the changes in the input variables of the obtained results is a critical step for E-IFKP portfolio problem.

The weights of the experts are of great importance on the results of the E-IFKP portfolio problem. A sensitivity analysis consisting of 8 different scenarios has been conducted to analyze the effect of the change in weight of each expert on the ranking results. In the analysis, a total of 8 different changes have been applied in the weights of the three experts included in the study, and the final results are different in the cases indicated. Based on the software final results in these cases, we can conclude that the optimal portfolio selections in the described cases differ. The results of the software show that there is sensitivity in the output results when including 1, 2 and 4 assets; or 1 and 4 assets; 2 and 4 assets. In some of these cases, the optimization problem is invalid from the point of view of IF logic.

A sensitivity analysis was performed by changing the input data by $\pm10\%, \pm25\%, \pm50\%$ and $\pm75\%$ respectively. In all these cases, the input data was invalid from the IF point of view.

IV. Conclusion

A 0-1 E-IFKP approach for portfolio selection was established in this study expanding 0-1 C-IFKP from [46] and the classical dynamic optimization algorithm for this problem [59]. The software being developed for the performance of the E-IFKP approach is applied to a real case for the selection of portfolio shares of the IT companies which make up the Dow Jones Industrial Average. Three scenarios are proposed to the decision maker for the final choice - pessimistic, optimistic, and average. Future research will include expanding this E-IFKP technique to three-dimensional intuitionistic fuzzy data [3] as well as developing software for its implementation.

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