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# Extension-principle-based Solution Algorithm to Full LR-fuzzy Linear Programming Problems

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***Abstract*—In the literature one can find various methods for solving full fuzzy linear programming problems. Very few of them fully comply to the extension principle. In the current study we extend an existing solution approach based on the extension principle to derive fuzzy-set optimal results to full fuzzy linear programming problems with L-R fuzzy parameters.**

**Our approach is a twofold extension of a procedure from the literature: (i) it employs L-R membership functions in the optimization models that derive fuzzy-set optimal objective values; and (ii) it introduces new optimization models for deriving fuzzy-set optimal solution values in accordance to the extension principle and product operator. The employment of the product operator makes the derived fuzzy-set results more thinner, thus more appropriate to further decision making.**

## I. Introduction

FUZZY SETS are currently used within the soft computing field to handle uncertainties when modeling real systems. Full fuzzy programming problems are widely addressed in the recent literature. The majority of the solution approaches involve cumbersome case analyses, and provide trivial solutions as soon as the appropriate case is identified. Many times, such solutions are non-consistent, since the applied methodologies do not comply to the extension principle (EP) [1].

Particularly, procedures for solving full fuzzy linear programming problems, that work in full accordance to the extension principle, were reported in the literature. The $\alpha$-cut intervals were used in all these procedures to derive the fuzzy set optimal solutions. The min operator firstly used within the extension principle to aggregate the parameters was later on replaced by the product operator in order to achieve thinner fuzzy set optimal solutions to the same problems [2].

An extension-principle-based methodology is available only for problems involving trapezoidal fuzzy parameters so far, thus the purpose of this study is to provide one for problems involving general L-R fuzzy parameters. The proposed methodology will extend the existing models for deriving the fuzzy set optimal values of the objective functions, and provide new optimization models to obtain the fuzzy set optimal solution values of the decision variables.

Firstly, the paper contributes to the enlargement of the class of full fuzzy optimization problems that can be solved numerically. Secondly, it provides more detailed solutions, in the sense of describing numerically the optimal values of the decision variables, not only of the objective function. Moreover, the solution approach we propose can be widely used to validate other methodologies capacity to derive solutions in accordance to the extension principle. The results presented in this paper have theoretical foundations proved mathematically, and numerical illustrations.

The rest of the paper is organized as follows: Section II briefly surveys the relevant literature; Section III explains all necessary notation and provides the basic terminology; Section IV presents our solution approach. An illustrative example is offered in Section V; and the final remarks are included in Section VI.

## II. Literature review

Zimmermann [3] was the first to apply the fuzzy set theory to mathematical programming. A wide survey of the papers from the literature providing various models and solutions to fuzzy linear programming problems can be found in [4]. Perez-Canedo, Verdegay and Concepcion-Morales [5] reviewed the recent approaches to fuzzy linear programming based on lexicographic methods.

Full fuzzy linear programming (FF-LP) problems refer to linear-programming-shape problems having fuzzy parameters and decision variables. Majority of papers addressing FF-LP problems a priori impose the type of fuzzy sets used for both parameters and variables; apply the fuzzy arithmetic via the extension principle; and then optimize a single or multiple crisp objective function to derive the final solution. For instance Kumar, Kaur and Singh [6] imposed equality constraints on the FF-LP model, and transformed it to a crisp optimization problem using a ranking function. Later on, Ezzati, Khorram and Enayati [7] used a lexicographic method to solve the multiple objective linear programming problem attached to the original FF-LP problem. Liu and Kao [8] introduced the solution concept to fuzzy transportation problems based on extension principle. Their approach was further adapted and improved in [9], [10].

Solutions to LP problems with trapezoidal fuzzy coefficients and using score functions were provided by Suriyapriya, Murugan, and Nayagam in [11]; and some advanced techniques to solving intuitionistic fuzzy multiple objective non-linear optimization problems were introduced by Rani, Ebrahimnejad, and Gupta in [12]. Perez-Canedo and Concepcion-Morales [13] and [14] addressed FF-LP problems with L-R fuzzy and intuitionistic fuzzy parameters and decision variables. Both their mathematical models had inequality constraints. In the second study unrestricted (free) variables were also considered. Our approach do not involve any comparison of L-R fuzzy numbers, thus avoiding any use of ranking functions and being applicable to solve any FF-LP model including any constraint types or variables. The first attempt to empirically solve FF-LP problems based on the extension principle and using the generalized "min" operator was made by Stanojevic and Nadaban [2]. They proved that by using the product operator within the extension principle more narrow fuzzy set solutions can be derived.

Summing up, the solving methods for full fuzzy optimization problems use either scalar or vector defuzzification approaches before optimization, or the extension principle without any defuzzification. Ranking functions are used to attain defuzzifications (see for instance [15] and [16]). Vector defuzzification methods use more ranking functions to transform a fuzzy optimization problem into a crisp multiple objective optimization problem. The methods that do not involve any defuzzification use $\alpha$-cuts, $\alpha \in [0,1]$, thus numerically construct the final fuzzy set solution. A summary of the literature review is provided in Table I.

## III. Preliminaries

The basic terminology indispensable to this study is related to L-R fuzzy membership functions (introduced by Dubois and Prade [20]) and the extension principle (introduced by Zadeh [21]).

A fuzzy set $\widetilde{A}$ of a universe $U$ is formally described by the set of pairs $\left\{\left(x, \mu_{\widetilde{A}}(x)\right) | x \in U\right\}$, where $\mu_{\widetilde{A}} : U \to [0,1]$ is the membership function that applied on any element $x \in U$ provides its degree of fuzziness.

A fuzzy number is a special case of a convex, normalized fuzzy set of the universe of real numbers. An L-R fuzzy number $\widetilde{w}^{LR}$, also refered as quadruple $\left(w^1, w^2, w^3, w^4\right)$, has a membership function of the following form:

$$\mu_{\widetilde{A}}(x) = \begin{cases} L\left(\frac{w^2-x}{w^2-w^1}\right), & x \le m, \\ R\left(\frac{x-w^3}{w^4-w^3}\right), & x \ge n, \\ 0, & , \end{cases}$$

where both $L, R$ are defined on the set of real numbers such that $\mu_{\widetilde{A}}(x) \in [0,1]$. For our numerical illustrations we successively use $\max\left\{0, 1-x^2\right\}$ and $\max\left\{0, 1-\sqrt{x}\right\}$ to define both $L$ and $R$ functions.

A general full fuzzy optimization problem

$$\max\left\{f\left(\widetilde{a}, \widetilde{x}\right) | g\left(\widetilde{c}, \widetilde{x}\right) \le 0\right\}$$

with $\widetilde{a}$ and $\widetilde{c}$ vectors of fuzzy parameters, and $\widetilde{x}$ vector of fuzzy decision variables can be solved via Zadeh's extension principle using the solution concept that computes the membership degree of the optimal solutions to the crisp optimization problem

$$\max\left\{f\left(a, x\right) | g\left(c, x\right) \le 0\right\} \tag{1}$$

using the membership degree of the parameters. Let $p$ denote the vector obtained by concatenating the vectors $a$ and $c$, i.e. $p = (a|c)$. Then, the membership degree of $p$ is defined as

$$\mu(p) = \min\left\{\mu_{\widetilde{p_i}}\left(p_i\right) | i = 1, \ldots, m\right\}, \tag{2}$$

where $m$ is the number of scalar parameters used in modeling the original optimization problem. Then, the membership degree $\mu_{\widetilde{z}}(z)$, of the crisp optimal value $z$ in the fuzzy set of the optimal values $\widetilde{z} = \max\left\{f\left(\widetilde{a}, \widetilde{x}\right) | g\left(\widetilde{c}, \widetilde{x}\right) \le 0\right\}$ was defined by

$$\sup\left\{\mu\left(p\right) | p = \left(a|c\right), z = \max\left\{f\left(a, x\right) | g\left(c, x\right) \le 0\right\}\right\} \tag{3}$$

if there exist the parameter vectors $a$ and $c$ such that the optimal value of Problem (1) equals to $z$, or $\mu_{\widetilde{z}}(z) = 0$, otherwise. Similarly, for each scalar decision variable $\widetilde{x}_h$ the membership degree $\mu_{\widetilde{x}_h}(x_h)$ is defined as

$$\sup\left\{\mu\left(p\right) | p = \left(a|c\right), x_h = \arg_h \max\left\{f\left(a, x\right) | g\left(c, x\right) \le 0\right\}\right\} \tag{4}$$

if there exist the parameter vectors $a$ and $c$ such that the $h$-th component of the optimal solution to Problem (1) is $x_h$, or $\mu_{\widetilde{x}_h}(x_h) = 0$, otherwise.

## IV. Proposed methodology

The mathematical model of a FF-LP problem with L-R fuzzy parameters is given in (5).

$$\begin{array}{lll} \max & \sum_{j=1}^{n} \widetilde{c}_j^{LR} \widetilde{x}_j, & \\ \text{s.t.} & \sum_{j=1}^{n} \widetilde{a}_{ij}^{LR} \widetilde{x}_j \preceq \widetilde{b}_i^{LR}, & i = \overline{1, m}, \\ & \widetilde{x}_j \succeq 0, & j = \overline{1, n}, \end{array} \tag{5}$$

where $\overline{1, n}$ stands for the set of natural numbers from 1 to $n$, i.e. $\{1, 2, \ldots, n\}$. This model has non-negative decision variables and inequality constraints but it can be considered general from the point of view of our solution approach: to derive the final solutions, our approach collects the optimal results of the crisp LP problems

TABLE I
FEATURES OF OUR APPROACH AND SEVERAL APPROACHES FROM THE LITERATURE

| | References | Coefficients | Variables | Constraints | Optimization criterion / method |
|---|---|---|---|---|---|
| [17] | Allahviranloo et al. (2008) | L-R | unrestricted | inequality | ranking function, scalar defuzzification |
| [6] | Kumar et al. (2011) | TFN | restricted | equality | ranking function, scalar defuzzification |
| [7] | Ezzati et al. (2015) | TFN | restricted | equality | lexicographic, vector defuzzification |
| [18] | Kaur & Kumar (2016) | L-R | unrestricted | equality | lexicographic, vector defuzzification |
| [13] | Perez-Canedo et al. (2019) | L-R | unrestricted | inequality | lexicographic, vector defuzzification |
| [19] | Stanojevic & Stanojevic (2020) | TFN | unrestricted | any | EP based, "min" op., empiric $z$ |
| [2] | Stanojevic & Nadaban (2023) | TFN | unrestricted | any | EP based, "prod" op., empiric $z$ |
| | Current study | L-R | unrestricted | any | EP based, "prod" op., numeric $x$ and $z$ |

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} c_j x_j, \\ \text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j \le b_i, \quad i=\overline{1,m}, \\ & x_j \ge 0, \qquad j=\overline{1,n}. \end{aligned} \tag{6}$$

Problem (6) is the crisp analog of (5) in the sense of having the same shape but crisp parameters; and any crisp LP problem can be converted to (6) without loosing its generality.

On the other side, Problem (5) cannot represent the general form for other approaches that perform inconsistent fuzzy arithmetic operations on its parameters and variables.

Let us denote by

$$X_{A,b} = \left\{ x \in R^m | A^T x \le b^T, x \ge 0 \right\} \tag{7}$$

the feasible set of Problem (6), and by

$$U_{A,c} = \left\{ u \in R^m | A^T u \ge c^T, u \ge 0 \right\} \tag{8}$$

the feasible set of the dual of Problem (6).

The following Theorem 4.1 presents the theoretical foundation of Algorithm 1.

*Theorem 4.1:* For any value $\alpha^*$ arbitrary fixed in the interval $[0,1]$, the left and right endpoints of the interval representing the $\alpha^*$-cut of the fuzzy set of optimal values to Model (5) are equal to the optimal values of Model (9)

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} c_j x_j, \\ \text{s.t.} \quad & \\ & \left(\prod_{i=1}^{m}\prod_{j=1}^{n}\delta_{ij}\right)\left(\prod_{i=1}^{m}\beta_i\right)\left(\prod_{j=1}^{n}\gamma_j\right) = \alpha^*, \\ & a_{ij}^3 L^{-1}_{\widetilde{a}_{ij}^{LR}}(\delta_{ij}) - a_{ij}^1 \le a_{ij} \le a_{ij}^4 R^{-1}_{\widetilde{a}_{ij}^{LR}}(\delta_{ij}) + a_{ij}^2, \\ & b_i^3 L^{-1}_{\widetilde{b}_i^{LR}}(\beta_i) - b_i^1 \le b_i \le b_i^4 R^{-1}_{\widetilde{b}_i^{LR}}(\beta_i) + b_i^2, \\ & c_j^3 L^{-1}_{\widetilde{c}_j^{LR}}(\gamma_j) - c_j^1 \le c_j \le c_j^4 R^{-1}_{\widetilde{c}_j^{LR}}(\gamma_j) + c_j^2, \\ & \delta_{ij}, \beta_i, \gamma_j \in [0,1], i=\overline{1,m}, j=\overline{1,n}, \\ & x \in X_{A,b}; \end{aligned} \tag{9}$$

and Model (10)

$$\begin{aligned} \min \quad & \sum_{i=1}^{m} b_i u_i, \\ \text{s.t.} \quad & \\ & \left(\prod_{i=1}^{m}\prod_{j=1}^{n}\delta_{ij}\right)\left(\prod_{i=1}^{m}\beta_i\right)\left(\prod_{j=1}^{n}\gamma_j\right) = \alpha^*, \\ & a_{ij}^3 L^{-1}_{\widetilde{a}_{ij}^{LR}}(\delta_{ij}) - a_{ij}^1 \le a_{ij} \le a_{ij}^4 R^{-1}_{\widetilde{a}_{ij}^{LR}}(\delta_{ij}) + a_{ij}^2, \\ & b_i^3 L^{-1}_{\widetilde{b}_i^{LR}}(\beta_i) - b_i^1 \le b_i \le b_i^4 R^{-1}_{\widetilde{b}_i^{LR}}(\beta_i) + b_i^2, \\ & c_j^3 L^{-1}_{\widetilde{c}_j^{LR}}(\gamma_j) - c_j^1 \le c_j \le c_j^4 R^{-1}_{\widetilde{c}_j^{LR}}(\gamma_j) + c_j^2, \\ & \delta_{ij}, \beta_i, \gamma_j \in [0,1], i=\overline{1,m}, j=\overline{1,n}, \\ & u \in U_{A,c}, \end{aligned} \tag{10}$$

respectively.

Within these models, the objective functions are optimized over the variables $a_{ij}$, $b_i$, $c_j$, $\delta_{ij}$, $\beta_i$, $\gamma_j$, $i=\overline{1,m}$, $j=\overline{1,n}$. To derive the left (right) endpoints of the $\alpha^*$-cut of the fuzzy-set optimal solutions to Problem (5), for any fixed index $j^* \in \{1,2,\dots,n\}$, and $\alpha^* \in [0,1]$, we introduce Model (11)

$$\begin{aligned} & \min(\max) x_{j^*}, \\ & \text{s.t.} \\ & z_{\alpha^*}^{\min} \le \sum_{j=1}^{n} c_j x_j \le z_{\alpha^*}^{\max}, \\ & \left(\prod_{i=1}^{m}\prod_{j=1}^{n}\delta_{ij}\right)\left(\prod_{i=1}^{m}\beta_i\right)\left(\prod_{j=1}^{n}\gamma_j\right) = \alpha^*, \\ & a_{ij}^3 L^{-1}_{\widetilde{a}_{ij}^{LR}}(\delta_{ij}) - a_{ij}^1 \le a_{ij} \le a_{ij}^4 R^{-1}_{\widetilde{a}_{ij}^{LR}}(\delta_{ij}) + a_{ij}^2, \\ & b_i^3 L^{-1}_{\widetilde{b}_i^{LR}}(\beta_i) - b_i^1 \le b_i \le b_i^4 R^{-1}_{\widetilde{b}_i^{LR}}(\beta_i) + b_i^2, \\ & c_j^3 L^{-1}_{\widetilde{c}_j^{LR}}(\gamma_j) - c_j^1 \le c_j \le c_j^4 R^{-1}_{\widetilde{c}_j^{LR}}(\gamma_j) + c_j^2, \\ & \delta_{ij}, \beta_i, \gamma_j \in [0,1], i=\overline{1,m}, j=\overline{1,n}, \\ & x \in X_{A,b}, \end{aligned} \tag{11}$$

where $z_{\alpha^*}^{\min}$ and $z_{\alpha^*}^{\max}$ are the optimal values obtained by solving Models (9) and (10) for the given value $\alpha^* \in [0,1]$.

Algorithm 1 formally describes our solution approach. Generally, the predifined values $\alpha_1^*, \alpha_2^*, \dots, \alpha_q^*$ are chosen to be equidistant in the interval $[0,1]$, $\alpha_1^* = 0$, $\alpha_q^* = 1$.

**Algorithm 1** Deriving the fuzzy set solutions

**Require:** the predefined $\alpha$-levels $\alpha_1^*, \alpha_2^*, \ldots, \alpha_q^*$, and the L-R fuzzy-parameter matrices $\widetilde{A}^{LR}$, $\widetilde{b}^{LR}$, $\widetilde{c}^{LR}$.

**Ensure:** $z^{\min}$, $z^{\max}$, $x^{\min}$, $x^{\max}$.

**for** $k = 1, q$ **do**

Set $\alpha^* = \alpha_k^*$, and derive $z_{\alpha_k^*}^{\min}$ and $z_{\alpha_k^*}^{\max}$ by solving (9) and (10), respectively.

**for** $h = 1, n$ **do**

Derive $x_{\alpha_k^*}^{\min}(h^*)$ and $x_{\alpha_k^*}^{\max}(h^*)$ as minimal and maximal values of the objective function in (11).

**end for**

Construct the vectors $x_{\alpha_k^*}^{\min} = \left(x_{\alpha_k^*}^{\min}(h^*)\right)_{h^*=\overline{1,n}}$ and $x_{\alpha_k^*}^{\max} = \left(x_{\alpha_k^*}^{\max}(h^*)\right)_{h^*=\overline{1,n}}$.

**end for**

Construct $z^{\min} = \left(z_{\alpha_k^*}^{\min}\right)^{k=\overline{1,q}}$, $z^{\max} = \left(z_{\alpha_k^*}^{\max}\right)^{k=\overline{1,q}}$, $x^{\min} = \left(x_{\alpha_k^*}^{\min}\right)^{k=\overline{1,q}}$, $x^{\max} = \left(x_{\alpha_k^*}^{\max}\right)^{k=\overline{1,q}}$.

### A. Particularities due to specific shapes of L-R functions

In what follows we discuss the particularities of Models (9), (10) and (11) in certain cases, with respect to the L-R functions that are commonly used. All particular models are polynomial, or equivalent to polynomial models.

*1) $L(x) = R(x) = \max\{0, 1 - x\}$:* This particular case corresponds to trapezoidal fuzzy numbers. The fuzzy set of the optimal values of the objective functions were empirically described in [19] using min operator; and numerically derived in [2] using product operator. The new introduced Model (11) adapted to the current case becomes

$$\begin{array}{ll} \min(\max) & x_{j^*}, \\ \text{s.t.} & \\ & z_{\alpha^*}^{\min} \le \sum_{j=1}^{n} c_j x_j \le z_{\alpha^*}^{\max}, \\ & \left(\prod_{i=1}^{m}\prod_{j=1}^{n}\delta_{ij}\right)\left(\prod_{i=1}^{m}\beta_i\right)\left(\prod_{j=1}^{n}\gamma_j\right) = \alpha^*, \\ & a_{ij}^3(1-\delta_{ij}) - a_{ij}^1 \le a_{ij} \le a_{ij}^4(1-\delta_{ij}) + a_{ij}^2, \\ & b_i^3(1-\beta_i) - b_i^1 \le b_i \le b_i^4(1-\beta_i) + b_i^2, \\ & c_j^3(1-\gamma_j) - c_j^1 \le c_j \le c_j^4(1-\gamma_j) + c_j^2, \\ & \delta_{ij}, \beta_i, \gamma_j \in [0,1], i = \overline{1,m}, j = \overline{1,n}, \\ & x \in X_{A,b}. \end{array} \tag{12}$$

*2) $L(x) = R(x) = \max\{1 - \sqrt{x}\}$:* In this case, the inverse functions of $L$ and $R$ are $L^{-1}(y) = R^{-1}(y) = (1-y)^2$, and the constraints on variables $a_{ij}$, $b_i$, $c_j$, $i = \overline{1,m}$, $j = \overline{1,n}$ become quadratic, i.e.

$$\begin{array}{l} a_{ij}^3(1-\delta_{ij})^2 - a_{ij}^1 \le a_{ij} \le a_{ij}^4(1-\delta_{ij})^2 + a_{ij}^2, \\ b_i^3(1-\beta_i)^2 - b_i^1 \le b_i \le b_i^4(1-\beta_i)^2 + b_i^2, \\ c_j^3(1-\gamma_j)^2 - c_j^1 \le c_j \le c_j^4(1-\gamma_j)^2 + c_j^2, \end{array} \tag{13}$$

*3) $L(x) = R(x) = \max\{1 - x^2\}$:* In this case, the inverse functions of $L$ and $R$ are $L^{-1}(y) = R^{-1}(y) = \sqrt{1-y}$,

TABLE II
FUZZY PARAMETERS OF PROBLEM (15)

| Objective function and RHS | Constraints matrix |
|---|---|
| $\widetilde{c}_1 = (6, 8, 8, 10)$ | $\widetilde{a}_{11} = (4.5, 5, 5, 5.5)$ |
| $\widetilde{c}_2 = (10, 12, 12, 14)$ | $\widetilde{a}_{21} = (5.75, 6, 6, 6.25)$ |
| $\widetilde{c}_3 = (0.75, 1, 1, 1.25)$ | $\widetilde{a}_{31} = (0.5, 1, 1, 1.25)$ |
| $\widetilde{b}_1 = (105, 150, 155, 207)$ | $\widetilde{a}_{12} = (4.5, 5, 5, 5.5)$ |
| $\widetilde{b}_2 = (102, 120, 125, 147)$ | $\widetilde{a}_{22} = (1.75, 2, 2, 2.25)$ |
| $\widetilde{b}_3 = (58, 100, 110, 148)$ | $\widetilde{a}_{32} = (3.75, 4, 4, 4.5)$ |

and the constraints on variables $a_{ij}$, $b_i$, $c_j$, $i = \overline{1,m}$, $j = \overline{1,n}$ become

$$\begin{array}{l} a_{ij}^3\sqrt{1-\delta_{ij}} - a_{ij}^1 \le a_{ij} \le a_{ij}^4\sqrt{1-\delta_{ij}} + a_{ij}^2, \\ b_i^3\sqrt{1-\beta_i} - b_i^1 \le b_i \le b_i^4\sqrt{1-\beta_i} + b_i^2, \\ c_j^3\sqrt{1-\gamma_j} - c_j^1 \le c_j \le c_j^4\sqrt{1-\gamma_j} + c_j^2. \end{array} \tag{14}$$

With the help of the transformations

$$\begin{array}{l} \eta_{ij} = \sqrt{1-\delta_{ij}}, \theta_i = \sqrt{1-\beta_i}, \\ \vartheta_j = \sqrt{1-\gamma_j}, i = \overline{1,m}, j = \overline{1,n}, \end{array}$$

constraints (14) become linear, and the degree of the polynomial in the second constraint of Models (9), (10) and (11) is doubled. As a consequence, the final equivalent constraint system

$$\begin{array}{l} z_{\alpha^*}^{\min} \le \sum_{j=1}^{n} c_j x_j \le z_{\alpha^*}^{\max}, \\ \left(\prod_{i=1}^{m}\prod_{j=1}^{n}(1-\eta_{ij}^2)\right)\left(\prod_{i=1}^{m}(1-\theta_i^2)\right)\left(\prod_{j=1}^{n}(1-\vartheta_j^2)\right) = \alpha^*, \\ a_{ij}^3\eta_{ij} - a_{ij}^1 \le a_{ij} \le a_{ij}^4\eta_{ij} + a_{ij}^2, \\ b_i^3\theta_i - b_i^1 \le b_i \le b_i^4\theta_i + b_i^2, \\ c_j^3\vartheta_j - c_j^1 \le c_j \le c_j^4\vartheta_j + c_j^2, \\ \eta_{ij}, \theta_i, \vartheta_j \in [0,1], i = \overline{1,m}, j = \overline{1,n}, \\ x \in X_{A,b}, \end{array}$$

is polynomial.

## V. ILLUSTRATIVE EXAMPLE

Perez-Canedo et al. [13] used the following fuzzy optimization problem (15) to illustrate their approach.

$$\begin{array}{ll} \max & \widetilde{c}_1\widetilde{x}_1 + \widetilde{c}_1\widetilde{x}_2 + \widetilde{c}_1\widetilde{x}_3 \\ \text{s.t.} & \widetilde{a}_{11}\widetilde{x}_1 + \widetilde{a}_{12}\widetilde{x}_2 + \widetilde{x}_3 = \widetilde{b}_1, \\ & \widetilde{a}_{21}\widetilde{x}_1 + \widetilde{a}_{22}\widetilde{x}_2 \le \widetilde{b}_2, \\ & \widetilde{a}_{31}\widetilde{x}_1 + \widetilde{a}_{32}\widetilde{x}_2 \le \widetilde{b}_3, \\ & \widetilde{x}_1, \widetilde{x}_2 \ge 0, \\ & \widetilde{x}_3 \text{ free variable.} \end{array} \tag{15}$$

The fuzzy parameters of Problem (15) are given in Table II. Problem (15) has both equality and inequality constraints; and both bounded and unbounded variables. These characteristics make it relevant to describe the generality of the solution approaches.

Perez-Canedo et al. solved four variants of this problem using: (i) $L(x) = R(x) = 1 - x$; (ii) $L(x) = R(x) = 1 - x^2$;

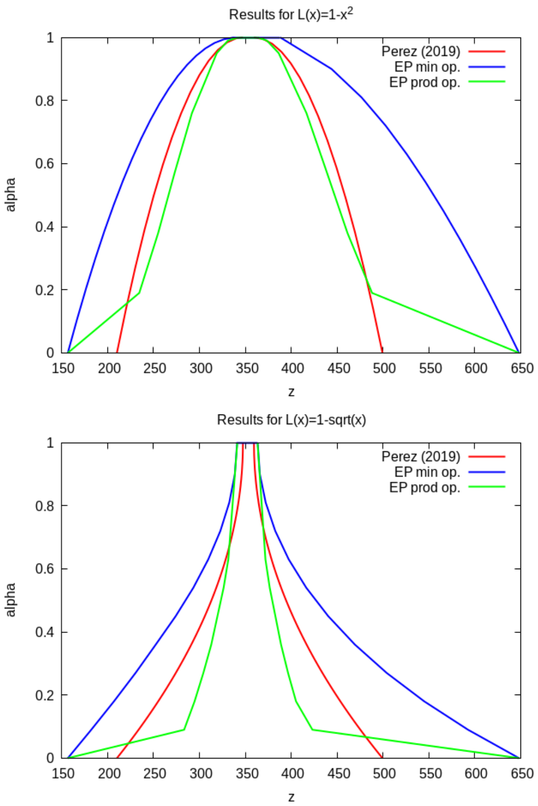


Fig. 1. Fuzzy set optimal values of the objective function

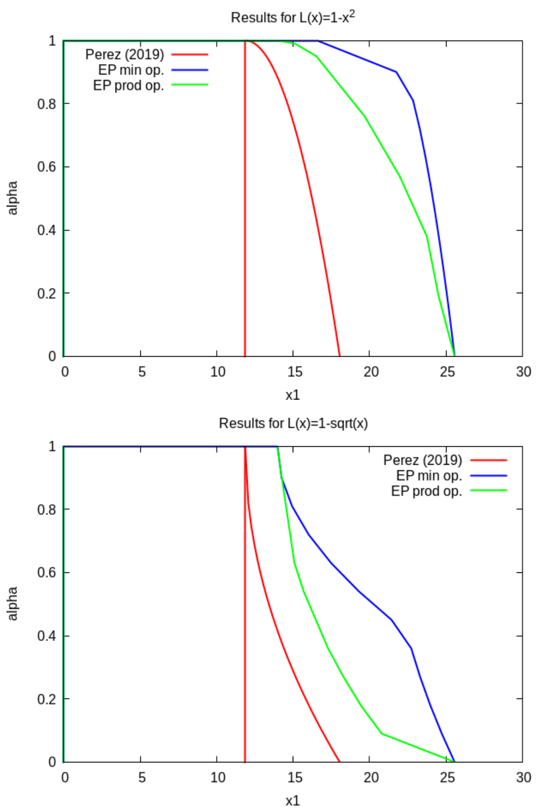


Fig. 2. Fuzzy set optimal values of variable $x_1$

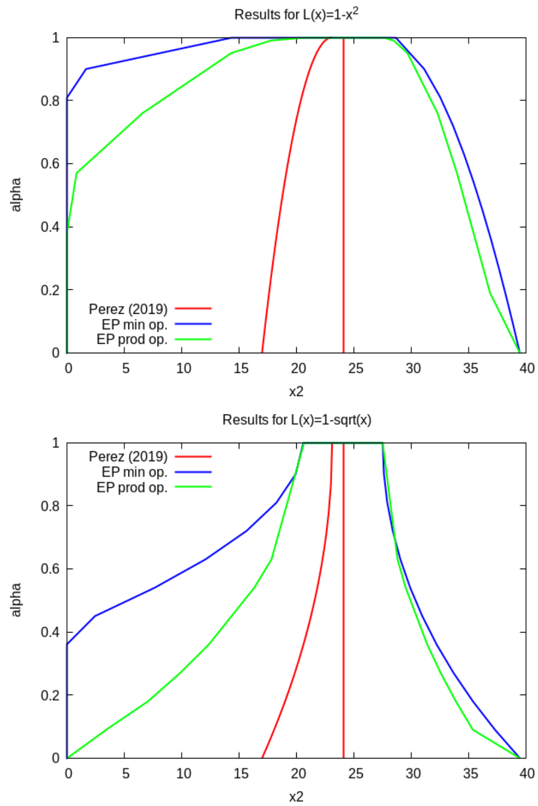


Fig. 3. Fuzzy set optimal values of variable $x_2$

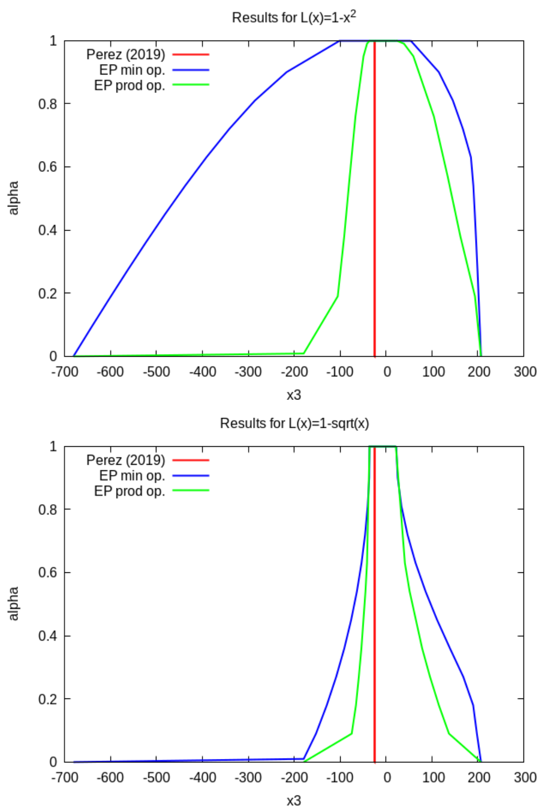


Fig. 4. Fuzzy set optimal values of variable $x_3$

(iii) $L(x) = R(x) = 1 - \sqrt{x}$; (iv) $L(x) = 1 - \sqrt{x}$ and $R(x) = 1 - x^2$.

The graphic representations obtained by applying our approach in comparison with the results obtained by Perez-Canedo et al. [13] are shown in Figures 1, 2 and 3. Figure 1 presents the fuzzy sets of the optimal objective values for both cases (ii) and (iii). Figures 2 and 3 present the fuzzy sets of the optimal solutions for the cases (ii) and (iii), respectively.

There are several facts that are well illustrated by this results: (i) the support of the fuzzy sets results (optimal objective and solution values) are not influenced by the operator "min" or "prod" used to aggregate the fuzzy parameters within extension principle, since whenever one parameter has membership degree 0, both "min" and "prod" operators provide the same membership value 0 after aggregation (ii) the shapes of the fuzzy set results (optimal objective and solution values) are much thinner when "prod" operator is used within extension-principle-based aggregation compared to "min" operator; (iii) the results obtained by Perez-Canedo et al. [13] are L-R fuzzy numbers and ours are not. Perez-Canedo et al. derived the support of the fuzzy sets results, and then applied the a priori imposed rule for obtaining the membership functions, while we derived the left and right endpoints of each desired $\alpha$-cut.

## VI. Conclusion and further researches

By this study we aimed to provide a solution algorithm to full fuzzy linear programming problems with L-R fuzzy descriptions to uncertain parameters. Comparing to other studies from the literature one of the advantages of our methodology is that it fully complies to the extension principle. In addition it uses the product operator to aggregate the L-R fuzzy quantities, thus deriving more narrow fuzzy set solutions to the original problem. Our approach introduced new optimization models for deriving fuzzy-set optimal solution values, and derives results that comply to the exstention principle. The proposed methodology is illustrated on a numerical example recalled from the literature. The class of fuzzy optimization problems that can be solved using similar principles can be further extended, e.g. nonlinear optimization problems in fuzzy environment can be addressed. Based on the same solution concept, an empiric variant of the new introduced approach - that simulates the extension principle by choosing randomly values of the parameters within their corresponding fuzzy sets - can be employed to estimate the fuzzy set solutions in accordance to the extension principle.

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