Proceedings of the 19th Conference on Computer Science and Intelligence Systems (FedCSIS) pp. 719–724 DOI: 10.15439/2024F3882 ISSN 2300-5963 ACSIS, Vol. 39

Secretary problem revisited: Optimal selection strategy for top candidates using one try in a generalized version of the problem

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***Abstract*—This paper explores a novel variation of the classical secretary problem, commonly referred to as the marriage or best choice problem. In this adaptation, a decision-maker sequentially dates $n \in \mathbb{N}$ candidates, each uniquely ranked without ties from 1 to n. The decision strategy involves a preliminary non-selection phase of the first $d \in \mathbb{N}$ candidates where, $d < n$, following which the decision-maker commits to the first subsequent candidate who surpasses all previously evaluated candidates in quality. The central focus of this study is the derivation and analysis of $P(d, n, k)$, which denotes the probability that the selected candidate, under the prescribed strategy, ranks among the top $k \in \mathbb{N}$ overall candidates, where $k \leq n$. This investigation employs combinatorial probability theory to formulate $P(d, n, k)$ and explores its behavior across various parameter values of d, n, and k. Particularly, we seek to determine in what fraction of the entire decision process should a decision-maker stop the non-selection phase, i.e., we search for the optimal proportion $\frac{d}{n}$, that maximizes the probability $P(d, n, k)$, with a special focus on scenarios where k is in generally low. While for $k = 1$, the problem is simplified to the classical secretary problem with $\frac{d}{n} \approx \frac{1}{e}$, our findings suggest that the strategy's effectiveness is optimized for portion $\frac{d}{n}$ decreasing below $\frac{1}{e}$ as k increases. Also, intuitively, probability $P(d, n, k)$ increases as k increases, since the number of acceptable top candidates increases. These results not only extend the classical secretary problem but also provide strategic insights into decision-making processes involving ranked choices, sequential evaluation, and applications of searching not necessarily the best candidate, but one of the best candidates.**

I. Introduction

THE secretary problem demonstrates a scenario involving optimal stopping theory, which is studied extensively in the fields of applied probability, statistics, and decision theory. Known under various names such as the marriage problem, the sultan's dowry problem, the fussy suitor problem, the googol game, and the best choice problem, its solution has garnered attention due to the intriguing nature of its simple yet effective decision strategy often referred to as the 37% rule [1].

In the classical form of the problem, an administrator aims to hire the best secretary out of n (uniquely) rankable applicants. Each applicant is interviewed one at a time in random order, with an immediate decision required at the end of each interview. Once rejected, an applicant cannot be recalled. The challenge lies in making a decision with incomplete information about the quality of unseen applicants, necessitating a strategy that balances the risk of passing up the best candidate against the potential for future superior candidates. The odds algorithm provides the shortest rigorous proof for this problem, establishing that the optimal win probability is always at least $\frac{1}{e}$, with the optimal stopping rule being to reject the first $\sim \frac{n}{e}$ applicants, i.e. roughly 37 % of n, and then stop at the first applicant who is better than all previously interviewed candidates.

Other modifications of the secretary problem explore various strategic nuances. One such variant is the "postdoc" problem [2], where the objective shifts from selecting the best to the second-best candidate (because the "best" will go to Harvard). Theoretical analysis shows that the success probability for this variant with an even number of applicants is exactly $\frac{0.25n^2}{n(n-1)}$, which simplifies to approximately $1/4$ as n grows large. This change underscores the subtlety needed in planning and execution when the goal is not to secure the top choice but a candidate just slightly less optimal.

Further expanding the range of strategic considerations, another version allows multiple selections, aiming to secure the top-k candidates using k tries [3]. Here, the challenge grows with k, as each choice potentially affects subsequent selections. Research indicates that the initial non-selection phase should last approximately $\left\lfloor \frac{n}{ke^{1/k}} \right\rfloor$ candidates, maximizing the chance of selecting all top-k candidates, which converges to $\frac{1}{ek}$ in probability as n becomes very large. In a sophisticated variant of the secretary problem, a decision-maker is granted multiple attempts to select the best candidate [4], each governed by a distinct set of r decision thresholds $(a_1, a_2, \ldots, a_r)$, where $a_1 > a_2 > \cdots > a_r$. As the number of interviews approaches infinity, the threshold for each decision-maker converges to ne^{-k_i}, where k_i is a defined constant [5]. Bruss and Louchard in [6] explored online selection strategies for choosing the κ best objects from n sequentially observed, rankable objects, with a focus on threshold functions that account for past selections and their asymptotic behavior as $n \to \infty$.

In this work, we extend the classic framework to develop and analyze $P(d, n, k)$, the probability that after skipping first d candidates, the candidate selected using a stopping rule, i.e., the one that got the highest ranking so far, ranks among the top k candidates out of n candidates in total. Our study examines the effects of varying the parameters d, n, and k to determine optimal selection strategy, particularly we search for proportion $\frac{d}{n}$ that maximizes probability $P(d, n, k)$. Besides different decision thresholds, we specifically explore the efficacy and the impact of increasing k on the strategy's performance. Skipping the first d candidates allows for establishing a robust benchmark of candidate quality without prematurely committing, thus striking a critical balance between not selecting too early (if d is small), which risks missing higher-quality candidates appearing later, and not starting too late (when d is large), which risks missing optimal candidates that have already been evaluated. This opens room for optimizing the value of d or $\frac{d}{n}$ which maximizes probability $P(d, n, k)$ of selecting one of the top k candidates.

II. A MODIFIED SECRETARY PROBLEM OF SEARCHING FOR TOP CANDIDATES

The classical secretary problem, also known as the marriage problem, traditionally focuses on identifying the optimal strategy to select the best candidate from a sequentially reviewed set. This section introduces a modified version of the secretary problem that expands the objective to include not just the best candidate but potentially any of the top few candidates, based on predefined criteria. This modification introduces a more complex and realistic scenario that decision-makers often encounter in various practical applications, from hiring processes to academic admissions.

A. Problem setup and notation

In this modified framework, a decision-maker sequentially dates $n \in \mathbb{N}$ candidates, each uniquely ranked from 1 to n without ties. The candidates are reviewed one at a time in a random sequence, and the decision-maker must decide immediately after each interview whether to select the candidate or continue with the process. Once a candidate is rejected, they cannot be recalled. The decision-maker initially evaluates d candidates without selecting any of them, where $1 \leq d < n$. This phase is crucial for establishing a benchmark against which all subsequent candidates are compared. It allows the decision-maker to gain a clear understanding of the average candidate quality, setting a standard that must be exceeded to initiate the selection phase. Starting from the $(d+1)$-th candidate and continuing with $(d+2)$-th, $(d+3)$-th candidate, ..., the decision-maker selects the first candidate who surpasses all previously evaluated candidates in quality. This approach aims to maximize the likelihood of choosing one of the top-ranked candidates by ensuring a thorough comparison to a well-established quality benchmark.

B. Description of the strategy

The specifics of the strategy can be further distilled into several key steps.

(i) Evaluate the first d candidates without selecting any (non-selection phase).
(ii) Begin the selection process with the $(d+1)$-th candidate.
(iii) Continue with $(d+2)$-th, $(d+3)$-th candidate, ..., and stop at the first candidate who is better than all the evaluated candidates so far.
(iv) If no such candidate is found by the end, either select the last candidate or leave the position unfilled, depending on specific rules which may be predefined in the problem statement.

This modified approach introduces a dynamic element to the decision-making process, where the decision-maker's strategy adapts based on the outcomes of initial evaluations. Let $P(d, n, k)$ be a probability that the selected candidate, using the prescribed strategy, is among the top k ranked candidates. The objective is to maximize the probability $P(d, n, k)$, which quantifies the success of the strategy in terms of selecting a top-ranked candidate. The subsequent section will delve into the solution methods and analytical techniques used to derive and maximize $P(d, n, k)$, offering insights into the optimal values of $\frac{d}{n}$ and the conditions under which the strategy succeeds.

C. Analytical derivation of $P(d, n, k)$

In this section, we delve into the analytical workings of the formula $P(d, n, k)$, which quantifies the probability that the selected candidate is among the top k out of n candidates, following a strategy where the first d candidates are merely evaluated and not selected, $1 \leq d < n$. The derivation involves considering each candidate i, where i ranges from $d+1$ to n, and calculating the probability that this i-th candidate is one of the k best.

Assume that the candidate i is being considered for selection, with $i \in \{d+1, d+2, \ldots, n\}$. The strategy entails skipping the initial d candidates, so the analysis starts from the $(d+1)$-th candidate. Several aspects ensure that the i-th candidate, who is selected, is one of top k candidates.

(i) To select i-th candidate, the decision-maker couldn't meet before any candidate rated higher than any of first

d candidates. Thus, the maximum rating among the first $i-1$ candidates (including the skipped d candidates) must lie within these d candidates, occurring with a probability of $\frac{d}{i-1}$.

(ii) The selected i-th candidate must be among the top k candidates in total.

- The probability that the i-th candidate is the absolute best among all n candidates is $\frac{1}{n}$.
- The probability that the i-th candidate is the second-best involves two conditions: first, the i-th position must actually be the second highest, which occurs with probability $\frac{1}{n}$, and second, the best candidate is not among the first i candidates but among those who follow (otherwise, they should be selected as a candidate with the highest ranking so far), which occurs with probability $\frac{n-i}{n-1}$. Therefore, the combined probability is
$$\frac{1}{n}\cdot\frac{n-i}{n-1}.$$
- For the i-th candidate to be the third-best, the logic extends further: the probability of being third is compounded by the likelihood that exactly two candidates ranked higher are still to come after candidate i, so equal to $\frac{\binom{n-i}{2}\cdot 2!}{\binom{n-1}{2}\cdot 2!}$. This probability is calculated as
$$\frac{1}{n}\cdot\frac{\binom{n-i}{2}\cdot 2!}{\binom{n-1}{2}\cdot 2!}=\frac{1}{n}\cdot\frac{(n-i)(n-i-1)}{(n-1)(n-2)}.$$
- Extending this to the general case for the j-th best, where $1\le j\le k$, the probability that the i-th candidate is the j-th best can be similarly modeled. It is the product of the probability that the i-th position is the j-th highest, $\frac{1}{n}$, and that all higher-ranked $j-1$ candidates appear among those $n-i$ yet to be seen, $\frac{\binom{n-i}{j-1}\cdot(j-1)!}{\binom{n-1}{j-1}\cdot(j-1)!}$, so

$$\begin{aligned}\frac{1}{n}\cdot\frac{\binom{n-i}{j-1}\cdot(j-1)!}{\binom{n-1}{j-1}\cdot(j-1)!}&=\frac{\binom{n-i}{j-1}\cdot(j-1)!}{n\cdot\frac{(n-1)!}{(j-1)!\,(n-j)!}\cdot(j-1)!}=\\&=\frac{\binom{n-i}{j-1}\cdot(j-1)!}{n\cdot\frac{(n-1)!}{(n-j)!}}=\\&=\frac{\binom{n-i}{j-1}\cdot(j-1)!}{\frac{n!}{(n-j)!}}=\\&=\frac{\binom{n-i}{j-1}\cdot(j-1)!}{\frac{n!}{(n-j)!\,j!}\cdot j!}=\\&=\frac{\binom{n-i}{j-1}\cdot(j-1)!}{\binom{n}{j}\cdot j!}\end{aligned}\tag{1}$$

The i-th selected candidate can be the first, the second, …, or the k-th best out of all n candidates, so $j\in\{1,2,\dots,k\}$. Since these states are obviously disjunctive, the probability that i-th selected candidate (for fixed $i\in\{d+1,d+2,\dots,n\}$) is among top k is, using formula (1), equals to

$$\sum_{j=1}^{k}\frac{\binom{n-i}{j-1}\cdot(j-1)!}{\binom{n}{j}\cdot j!}.\tag{2}$$

Since conditions (i) and (ii) from previous analysis are independent (condition (i) deals with arrangement of first $i-1$ candidates while condition (ii) handles last $n-i+1$ candidates sequence), we can derive that the probability for the i-th candidate being among the top k and better than the previous maximum observed among the first $i-1$ candidates can be represented (for fixed $i\in\{d+1,d+2,\dots,n\}$) as

$$\frac{d}{i-1}\sum_{j=1}^{k}\frac{\binom{n-i}{j-1}\cdot(j-1)!}{\binom{n}{j}\cdot j!},\tag{3}$$

and since i-th candidate can be $(d+1)$-th, $(d+2)$-th, …, n-th one (which are disjunctive), we get the complete probability that the i-th candidate is among the top k and with the highest ranking so far out of the first $i-1$ candidates, as follows

$$P(d,n,k)\overset{(1,2,3)}{=}\sum_{i=d+1}^{n}\frac{d}{i-1}\sum_{j=1}^{k}\frac{\binom{n-i}{j-1}\cdot(j-1)!}{\binom{n}{j}\cdot j!}.\tag{4}$$

III. Optimal Strategy for the Modified Secretary Problem of Searching for Top Candidates

We assume that a decision-maker knows or pre-estimates the number n of all candidates and also decides how many top candidates k are relevant for their selection. However, the decision-maker would like to know, at which candidate to stop only evaluate rating and start possible selecting to maximize probability $P(d,n,k)$ of selecting one of top k candidates. In other words, the decision-maker would like to know the value of d, or value of $\frac{d}{n}$.

A. Maximizing $P(d,n,k)$ for fixed $n\in\mathbb{N}$ and $k=1$ with respect to $1\le d<n$

Let's focus on simplifying the formula for $P(d,n,k)$ when $k=1$. This specific case indeed reverts the problem to the classical secretary problem [1], where the goal is to maximize the probability of selecting the best candidate out of n. For $k=1$, formula (4) simplifies significantly,

$$\begin{aligned}P(d,n,1)&=\sum_{i=d+1}^{n}\frac{d}{i-1}\sum_{j=1}^{1}\frac{\binom{n-i}{j-1}\cdot(j-1)!}{\binom{n}{j}\cdot j!}=\\&=\sum_{i=d+1}^{n}\frac{d}{i-1}\frac{\binom{n-i}{1-1}\cdot(1-1)!}{\binom{n}{1}\cdot 1!}=\\&=\sum_{i=d+1}^{n}\frac{d}{i-1}\frac{\binom{n-i}{0}\cdot(0)!}{\binom{n}{1}\cdot 1!}=\\&=\sum_{i=d+1}^{n}\frac{d}{i-1}\frac{1}{n}=\frac{d}{n}\sum_{i=d+1}^{n}\frac{1}{i-1}=\\&=\frac{d}{n}\sum_{i=d}^{n-1}\frac{1}{i}.\end{aligned}\tag{5}$$

Formula (5) can be simplified using harmonic series, where $H_n = \sum_{i=1}^{n} \frac{1}{i}$. Therefore, the partial sum from d to $n-1$ as in formula (5) can be expressed as

$$\sum_{i=d}^{n-1} \frac{1}{i} = \sum_{i=1}^{n-1} \frac{1}{i} - \sum_{i=1}^{d-1} \frac{1}{i} = H_{n-1} - H_{d-1},$$

thus, we improve formula (5) as

$$P(d,n,1) = \frac{d}{n} \sum_{i=d}^{n-1} \frac{1}{i} = \frac{d}{n}(H_{n-1} - H_{d-1}). \qquad (6)$$

For large n, the harmonic number H_n can be approximated using the natural logarithm as $H_n \approx \ln(n) + \gamma$, where γ is the Euler-Mascheroni constant. Applying this to our partial sum in formula (6), we get

$$\begin{aligned} P(d,n,1) &= \frac{d}{n}(H_{n-1} - H_{d-1}) \approx \\ &\approx \frac{d}{n}(\ln(n-1) + \gamma - (\ln(d-1) + \gamma)) \approx \\ &\approx \frac{d}{n}(\ln(n-1) - \ln(d-1)) \approx \\ &\approx \frac{d}{n} \cdot \ln\left(\frac{n-1}{d-1}\right). \end{aligned} \qquad (7)$$

Let's take a derivative of $P(d,n,1)$ from formula (7) with respect to d searching for the value of d maximizing $P(d,n,1)$,

$$\begin{aligned} \frac{\partial}{\partial d} P(d,n,1) &\approx \frac{\partial}{\partial d} \left\{ \frac{d}{n} \cdot \ln\left(\frac{n-1}{d-1}\right) \right\} = \\ &= \frac{1}{n} \cdot \ln\left(\frac{n-1}{d-1}\right) + \frac{d}{n} \frac{1}{\frac{n-1}{d-1}} \left(-\frac{n-1}{(d-1)^2}\right) = \\ &= \frac{1}{n} \cdot \ln\left(\frac{n-1}{d-1}\right) - \frac{d}{n(d-1)}. \end{aligned} \qquad (8)$$

Putting derivative $\frac{\partial}{\partial d} P(d,n,1)$ from formula (8) equal zero, we get

$$\begin{aligned} \frac{\partial}{\partial d} P(d,n,1) &\equiv 0 \\ \frac{1}{n} \cdot \ln\left(\frac{n-1}{d-1}\right) - \frac{d}{n(d-1)} &= 0 \\ \frac{1}{n} \cdot \ln\left(\frac{n-1}{d-1}\right) &= \frac{d}{n(d-1)} \\ \ln\left(\frac{n-1}{d-1}\right) &= \frac{d}{d-1}, \end{aligned}$$

and for large n and d is $\frac{n-1}{d-1} \approx \frac{n}{d}$ and $d \approx d-1$, so

$$\begin{aligned} \ln\left(\frac{n-1}{d-1}\right) &= \frac{d}{d-1} \\ \ln\left(\frac{n}{d}\right) &\approx \frac{d}{d} = 1, \end{aligned} \qquad (9)$$

which results in $\frac{n}{d} \approx e$ or $\frac{d}{n} \approx \frac{1}{e}$, where e is Euler constant. The well-known 37% rule comes from the equation $\frac{d}{n} \approx \frac{1}{e}$ in formula (9) since $\frac{1}{e} \approx 0.369$.

B. Maximizing $P(d,n,k)$ for fixed $n \in \mathbb{N}$ and $k > 1$ with respect to $1 \leq d < n$

To simplify and analyze $P(d,n,k)$ in a continuous manner which would enable us to search for $\frac{d}{n}$ maximizing $P(d,n,k)$, we consider a transformation using continuous approximations. Assuming a large n, the sums can be approximated by integrals,

$$P(d,n,k) \approx \int_{d}^{n} \frac{d}{x-1} \sum_{j=1}^{k} \left(\frac{1}{j} \cdot \frac{\binom{n-x}{j-1}}{\binom{n}{j}} \right) \mathrm{d}x. \qquad (10)$$

Given the transcendental nature of the expressions in formula (10), involving exponential and logarithmic functions in integral form, and the eventual use of the gamma function $\Gamma(x) = (x-1)!$ in place of combinatorial numbers, numerical methods are preferred for solving the optimal parameters, highlighting the complexity and non-linearity of the problem.

C. Maximizing $P(d,n,k)$ for fixed $n \in \mathbb{N}$ and $1 \leq k \leq n$ with respect to $1 \leq d < n$ using numerical approaches

Formula (10) indicates that a numerical searching for $\frac{d}{n}$ that maximizes $P(d,n,k)$ for fixed $n \in \mathbb{N}$ and $1 \leq k \leq n$, where d is any value in $1 \leq d < n$, could be more promising than an analytical solution.

We set $n = 20$ and searched numerically for $\frac{d}{n}$ that maximizes $P(d,n,k)$ for $k \in \{1, 2, \ldots, 20\}$. We used both formula (4) and also a function `getMyProbability`(n,d,k,m) based on Monte Carlo simulation, see Algorithm 1, that estimates $P(d,n,k)$ probability for $m = 50$ random samples of candidates for each combination of values (d,n,k). The outcomes of function `getMyProbability`(n,d,k,m) should confirm analytical correctness of formula (4) for $P(d,n,k)$ probability. In Algorithm 1, we set $d < n$ because if $d = n$, the selection phase cannot occur, as it begins with the $(d+1)$-th candidate. We denote the probability as $\hat{P}(d,n,k)$ in Algorithm 1 instead of $P(d,n,k)$, as the algorithm uses a finite number (m) of simulated repetitions. If Algorithm 1 is repeated t times, yielding $\hat{P}(d,n,k)_\tau$ in its τ-th iteration, the following relationship holds, $P(d,n,k) = \lim_{t \to \infty} \frac{1}{t} \sum_{\tau=1}^{t} \hat{P}(d,n,k)_\tau$.

In Fig. 1, we see the values of $\frac{d}{n}$ for $n = 20$ and $k \in \{1, 2, \ldots, 8\}$. While the boxplots come from simulated samples of candidates so that function `getMyProbability`(n,d,k,m) estimated value of $P(d,n,k)$ across all $m = 50$ repetitions for a given combination of (d,n,k), the blue line shows analytically computed values of $P(d,n,k)$ using formula (4).

As we expected, the proportion $\frac{d}{n}$ decreases as k increases – it is feasible, if more candidates are acceptable for selection (larger k), the more of them are likely in last $n-d$ candidates in a row, thus, d and $\frac{d}{n}$ could diminish. As k increases, the probability $P(d,n,k)$ of selecting a top k candidate rises, which is intuitive since accepting more candidates increases the chances of selection. Optimal values of d^* and $\frac{d^*}{n}$ that maximizes $P(d,n,k)$ for $n = 20$ and $k \in \{1, 2, \ldots, 20\}$, as

Algorithm 1: Estimation of probability $P(d, n, k)$ of selecting one of top k candidates

Function `getMyProbability`(n, d, k, m)**:**
- **Input :** n (total # of candidates), d (# of candidates to meet without selecting), k (# of top ranked candidates), m (# of scenarios to simulate)
- **Output:** estimate of $P(d, n, k)$ probability
- $c \leftarrow 0$;
- **for** $i \leftarrow 1$ **to** m **do**
  - `sample` $\leftarrow$ sample integers from 1 to n without replacement;
  - **if** $d < n$ **then**
    - $j \leftarrow d + 1$;
    - **while** $j \leq n$ *and* $sample[j] < \max(sample[1 : d])$ **do**
      - $j \leftarrow j + 1$;
    - **end**
    - **if** $j \leq n$ *and* $sample[j]$ *is in the top* k *positions of* n **then**
      - $c \leftarrow c + 1$;
    - **end**
  - **end**
- **end**
- $\hat{P}(d, n, k) \leftarrow \frac{c}{m}$;
- **return** $\hat{P}(d, n, k)$;

return

TABLE I
OPTIMAL VALUES OF d^* AND RATIOS $\frac{d^*}{n}$ THAT MAXIMIZES PROBABILITY $P(d^*, n, k)$, AND MAXIMUM VALUE OF THE PROBABILITY FOR SELECTING ONE OF THE TOP k CANDIDATES OUT OF $n = 20$. THE PROPORTION $\frac{d^*}{n}$ DECREASES AS k INCREASES, INDICATING A SHIFT TOWARDS EARLIER SELECTION FOR LESS RESTRICTIVE OUTCOMES.

k	d^*	$\frac{d^*}{n}$	$P(d^*, n, k)$
1	7	0.350	0.384
2	6	0.300	0.538
3	5	0.250	0.627
4	4	0.200	0.687
5	4	0.200	0.730
6	3	0.150	0.766
7	3	0.150	0.794
8	3	0.150	0.813
9	2	0.100	0.836
10	2	0.100	0.856
11	2	0.100	0.870
12	2	0.100	0.881
13	1	0.050	0.889
14	1	0.050	0.907
15	1	0.050	0.922
16	1	0.050	0.934
17	1	0.050	0.942
18	1	0.050	0.947
19	1	0.050	0.950
20	1	0.050	0.950

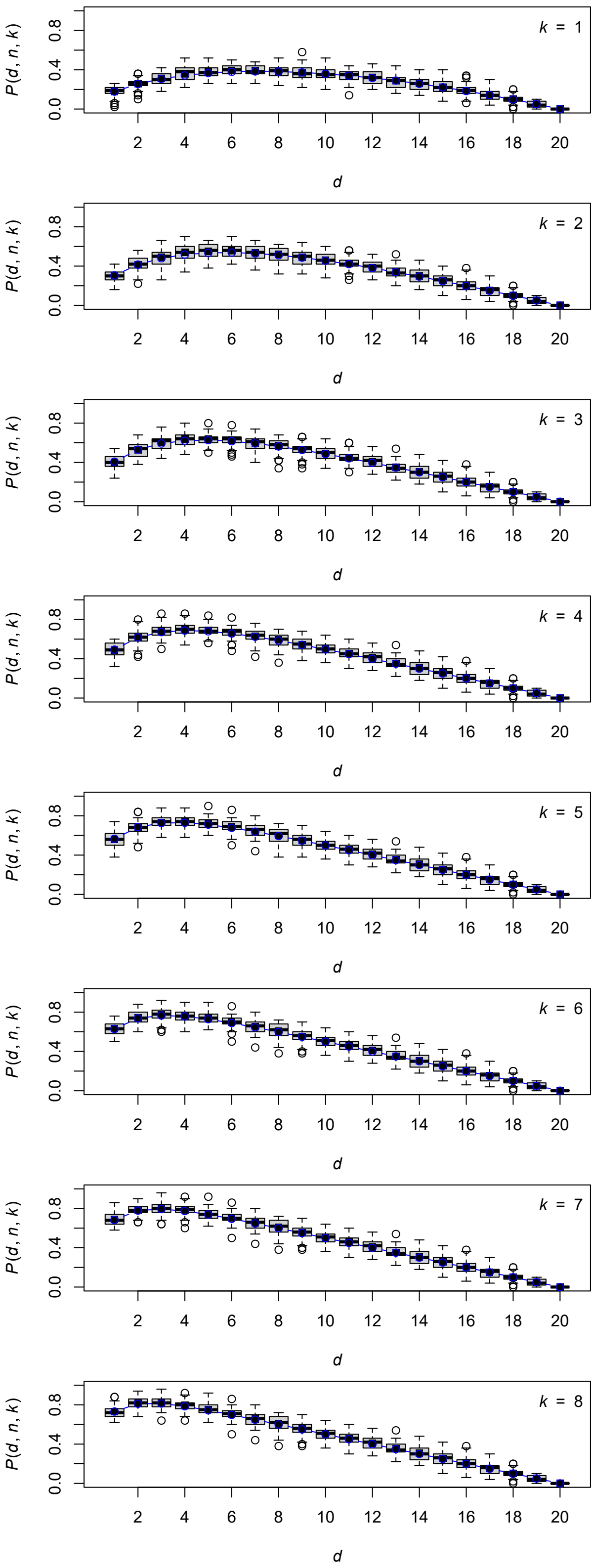

Fig. 1. Optimization of $\frac{d}{n}$ for maximizing $P(d, n, k)$ across k values from 1 to 20 with $n = 20$, using both analytical formula (4) and Monte Carlo simulations via `getMyProbability`(n, d, k, m) (Algorithm 1). As k increases, $P(d, n, k)$ rises while $\frac{d}{n}$ decreases, demonstrating a trade-off in selection strategy efficiency. Boxplots come from $m = 50$ random repetitions of sample generation for given combinations of (d, n, k) and confirm the analytical model's predictions. Blue line stands for values of analytically calculated $P(d, n, k)$.

well as the maximized values of probability $P(d^*, n, k)$, are in Table I, and correspond to outcomes from Fig. 1.

Finally, in Fig. 2, there are values of $\frac{d^*}{n}$ that maximize $P(d, n, k)$ for $n = 100$ and $k \in \{1, 2, \ldots, 100\}$, and Fig. 3 shows maximized values of $P(d^*, n, k)$ for varying values $k \in \{1, 2, \ldots, 100\}$ and $n = 100$. Similarly as before, maximizing proportions $\frac{d^*}{n}$ decrease and maximized probabilities $P(d^*, n, k)$ increase, as k increases. Both Table I and Fig. 2 show that for a restrictive approach when only top $k = 3$ are acceptable for selection, we need to stop the non-selecting phase not earlier than before skipping first 25 % of candidates, i.e., $\frac{d^*}{n} \approx 0.25$, which could be called as a 25% rule.

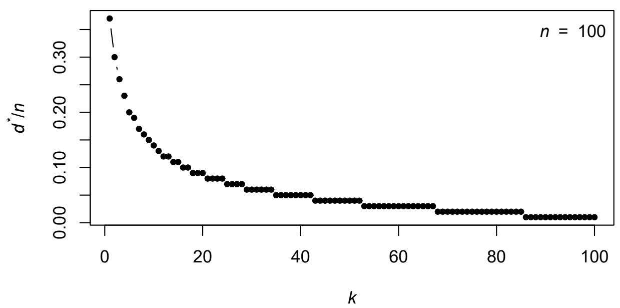

Fig. 2. Values of $\frac{d^*}{n}$ that maximize $P(d, n, k)$ for $n = 100$ and varying values $k \in \{1, 2, \ldots, 100\}$.

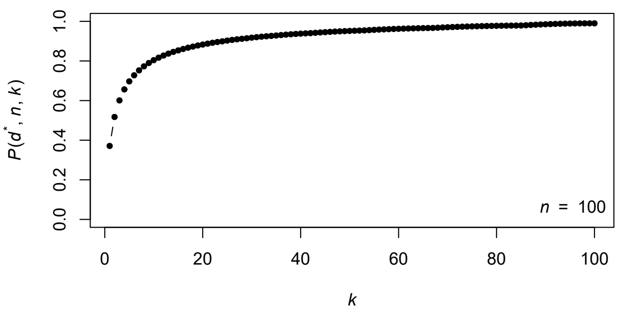

Fig. 3. Maximized values of $P(d^*, n, k)$ for $n = 100$ and varying values $k \in \{1, 2, \ldots, 100\}$.

IV. Conclusion

This study has extended the classical secretary problem by exploring the probability $P(d, n, k)$ of successfully selecting one of the top k candidates after initially skipping d candidates. Using probability and combinatorics, we illustrated that analytical methods for searching values of proportions $\frac{d}{n}$ maximizing probability $P(d, n, k)$ are complex, thus, we use Monte Carlo simulations to investigate the optimal strategies for different settings of k.

Our findings indicate a clear strategy shift depending on the rank acceptability, k. As k increases, allowing for a less restrictive choice, the optimal $\frac{d}{n}$ ratio decreases, signifying that an earlier selection becomes preferable. For $k = 1$, which reflects the traditional secretary problem, the optimal skipping strategy is $\frac{d}{n} \approx \frac{1}{e} \approx 0.369$, consistent with the well-known theoretical result of skipping the first 37 % of candidates. Also, probability $P(d, n, k)$ rises as k increases – intuitively, the more candidates are acceptable for selection, the higher is the chance for selecting one of them. When considering only the top $k = 3$ candidates for selection, the non-selecting phase should extend through at least the first 25 % of candidates, effectively establishing a 25% rule.

For real-world applications, such as hiring processes or competitive selection scenarios, these insights can guide more nuanced and more practical strategies that balance risk and reward effectively according to the range of acceptable outcomes. Also, in various applications, we do not want necessarily select the very best candidate, but one of top candidates is enough.

V. Acknowledgment

This paper is supported by the grant IG410023 with no. F4/50/2023, which has been provided by the Internal Grant Agency of the Prague University of Economics and Business.

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