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Improved upper bounds on the shortest watchman route in simple polygons: Dependence on reflex vertices and triangulation strategies

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***Abstract*—This paper investigates upper bounds on the length of the shortest watchman route in simple polygons using minimal structural information – specifically, the number and placement of reflex vertices – rather than full geometric detail. We show that the set of reflex vertices alone suffices to guarantee full visibility of the polygon: every interior or boundary point is visible from at least one reflex vertex. Moreover, these vertices can always be connected by a polyline within the polygon, enabling a constructive approach to route design. While it is known that the shortest watchman route is bounded above by the perimeter of the polygon, we prove that this bound is tight and cannot be improved in general. However, we identify several important classes of polygons where significantly better bounds are achievable. In particular, for polygons with exactly two reflex vertices, we establish that the shortest watchman route is strictly shorter than half the perimeter. We also introduce a novel triangulation method – *successive dual-edge triangulation* – in which each triangle is constructed by attaching to existing polygon edges. When such a triangulation is possible, the resulting watchman route has length strictly less than half the perimeter, regardless of the number of sides or reflex vertices. These results are not only theoretical but also constructive, offering efficient route designs when geometric data is available. The approach is particularly well suited to real-world environments such as industrial halls or private courtyards, where structural simplicity often permits route planning without requiring full geometric reconstruction. In such cases, our methods provide a significant and elegant reduction in route length compared to perimeter-based patrolling, making them applicable to scenarios such as autonomous drone navigation and indoor surveillance.**

I. Introduction

THE watchman route problem is a fundamental question in computational geometry and visibility theory. It seeks the shortest path within a polygonal domain from which every point is visible, with applications in robotic surveillance, patrolling, and security optimization.

For simple polygons (without holes), several polynomial-time algorithms exist. The first general solution, by Carlsson et al. [1], had worst-case complexity $\mathcal{O}(n^6)$, later improved to $\mathcal{O}(n^5)$ by Tan [2], and to $\mathcal{O}(n^4 \log\log n)$ in special cases by Chin and Ntafos [3], where n is the number of vertices. These methods assume full geometric detail, making them less practical when only structural features are known.

In polygons with holes, Mitchell showed the problem is NP-hard and proposed the first polynomial-time approximation algorithm with an $\mathcal{O}(\log^2 n)$ factor and showed strong inapproximability results [4]. Dumitrescu and Tóth [5] provided asymptotic upper bounds on the watchman route length in polygons with holes, expressed as $\mathcal{O}\left(\operatorname{perim}(\mathcal{P}) + \sqrt{k}\cdot\operatorname{diam}(\mathcal{P})\right)$ for a polygon $\mathcal{P}$ with k holes, perimeter $\operatorname{perim}(\mathcal{P})$, and diameter $\operatorname{diam}(\mathcal{P})$. However, they did not derive explicit route lengths or provide exact constants in the upper bounds. In contrast, this paper provides exact and constructive bounds for specific polygon classes.

Extensions of the problem include limited-visibility models [6], [7], multi-watchman strategies [8], and static coverage ratios from art gallery results [9], [10]. Similar geometric reasoning appears in other domains such as bioinformatics. For example, Lavor et al. [11] solve the discretizable molecular distance geometry problem using partial distance constraints and structural sparsity to infer configurations.

Most existing approaches to the watchman route problem focus on minimizing algorithmic complexity or improving approximation ratios, typically assuming full geometric data.

In practice, however, environments such as industrial halls or courtyards often have simple layouts, where detailed mapping is unnecessary and only coarse information – such as the number of vertices or reflex angles – is available.

This paper studies the watchman route problem in simple polygons under such limited assumptions. We show that the set of reflex vertices alone suffices to ensure full visibility, and that these vertices can always be connected by a polyline. The shortest watchman route is thus bounded above by the polygon's perimeter, and we prove that this bound is tight in general.

We then identify special structural cases that allow for strictly better bounds. For example, if the polygon has exactly two reflex vertices, the length of the shortest route is strictly less than half the perimeter. We also introduce a geometric method – *successive dual-edge triangulation* – in which the polygon is incrementally triangulated by attaching new triangles to existing edges. When such a triangulation is possible, the route length is guaranteed to be strictly less than half the perimeter, regardless of the polygon's size or complexity.

While our results are derived from general structural principles, they also yield constructive algorithms. These are well suited to practical scenarios where detailed geometry is unavailable, and even when such data is present, our method offers an elegant and analytically grounded alternative to exhaustive geometric search.

II. Preliminaries

This section introduces the geometric foundations, mostly elementary but essential, that underlie our results. We define point visibility and show that every point in a simple polygon is visible from some boundary point. In convex polygons, all pairs of points are mutually visible. For polygons with reflex vertices, each point is visible from at least one reflex vertex. Finally, we show that all reflex vertices can be connected by a polyline within the polygon, a key step in the proposed constructing watchman routes.

Definition 1 (Visibility of a point by another)**.** *Let $\mathcal{P}$ be a simple polygon and let $A \in \mathcal{P}$, $B \in \mathcal{P}$ be two points (either in the interior or on the boundary of $\mathcal{P}$). We say that* point A is visible by point B *(or equivalently,* B is visible from A*) if the closed line segment $\overline{AB}$ lies entirely within $\mathcal{P}$, that is, $\overline{AB} \subset \mathcal{P}$. Equivalently, we may say that* point A sees point B*, or vice versa.*

Lemma 1 (Visibility from the boundary of polygon)**.** *Let $\mathcal{P}$ be a simple polygon (of a finite size). Then every point $A \in P$ (including points in the interior and on the boundary) is visible by at least one point $B \in \partial\mathcal{P}$, where $\partial\mathcal{P}$ denotes the boundary of the polygon.*

Proof. Obviously $\partial\mathcal{P} \subset \mathcal{P}$ (†). Let $A \in P$ be an arbitrary point.

If A lies on the boundary $\partial\mathcal{P}$, i.e. $A \in \partial\mathcal{P}$, then it is trivially visible by itself, so $B = A$, because

$$\overline{AB} = B = A \in \partial\mathcal{P} \overset{(\dagger)}{\subset} \mathcal{P}.$$

Suppose A lies in the interior of $\mathcal{P}$. Since $\mathcal{P}$ is a simple polygon of a finite size, the line segment from A to any direction will eventually intersect the boundary of $\mathcal{P}$. Let us select an arbitrary direction and consider the ray starting at A in that direction. Let C be the first point where this ray intersects the boundary $\partial\mathcal{P}$, so $C \in \partial P$. Since $\mathcal{P}$ is simple and of a finite size (i.e., it has no self-intersections and encloses a connected finite region), the point C exists, segment $\overline{AC}$ has $|\overline{AC}| < \infty$ and lies entirely within $\mathcal{P}$, i.e., $\overline{AC} \subset \mathcal{P}$ (‡). Hence, $B = C$ and A is visible by B because

$$\overline{AB} = \overline{AC} \overset{(\ddagger)}{\subset} \mathcal{P}.$$

Thus, in all cases, there exists a boundary point $B \in \partial\mathcal{P}$ such that A is visible by B. ☐

Lemma 2 (Mutual visibility in convex polygons)**.** *Let $\mathcal{P}$ be a simple and convex polygon. Then every pair of distinct points $A \in \mathcal{P}$, $B \in \mathcal{P}$ (including interior and boundary points) are mutually visible; that is, the segment $\overline{AB}$ lies entirely within $\mathcal{P}$.*

Proof. Let $A, B \in \mathcal{P}$ be two distinct points, each either in the interior or on the boundary of the convex polygon $\mathcal{P}$. Consider the line $\overleftrightarrow{AB}$ passing through points A and B. Since $\mathcal{P}$ is convex and of finite size, the intersection of this line with the polygon $\mathcal{P}$ is a single line segment $\overline{CD}$, where $C, D \in \partial\mathcal{P}$ are the boundary points where the line enters and exits the polygon. Because $\mathcal{P}$ is convex, the entire segment $\overline{CD}$ lies within $\mathcal{P}$. Since both A and B lie on the segment $\overline{CD}$, so $\overline{AB} \subseteq \overline{CD}$, it follows that $\overline{AB} \subseteq \mathcal{P}$. Therefore, $\overline{AB} \subseteq \mathcal{P}$, which implies that A is visible by B and vice versa. ☐

Theorem 1 (Visibility from reflex vertices)**.** *Let $\mathcal{P}$ be a simple polygon with boundary $\partial\mathcal{P}$, and suppose that $\mathcal{P}$ contains at least one vertex $V \in \partial\mathcal{P}$ with a reflex angle. Then every point $A \in \mathcal{P}$ (including interior and boundary points) is visible by at least one reflex vertex of $\mathcal{P}$.*

Proof. Let $A \in \mathcal{P}$ be an arbitrary point. Let $\mathcal{W}_A$ denote the set of all boundary points visible from A, that is,

$$\mathcal{W}_A = \left\{\forall B \in \partial\mathcal{P} : \overline{AB} \subset \mathcal{P}\right\}.$$

Case 1: Suppose A sees the entire boundary, i.e., $\mathcal{W}_A = \partial\mathcal{P}$. Since $\mathcal{P}$ contains at least one reflex vertex V, it follows that $V \in \mathcal{W}_A$, and therefore $\overline{AV} \subset \mathcal{P}$. Hence, A is visible by a reflex vertex V.

Case 2: Suppose A does not see the entire boundary, i.e., $\mathcal{W}_A \subset \partial\mathcal{P}$. Let $C \in \partial\mathcal{P} \setminus \mathcal{W}_A$ be a boundary point not visible from A.

Subcase 2a: If $\mathcal{W}_A$ contains at least one reflex vertex V, then A is visible by V, and the statement holds.

Subcase 2b: Suppose $\mathcal{W}_A$ contains no reflex vertices; that is, every boundary point visible from A is either a convex vertex or a point on an edge between convex vertices. Since C is not visible from A and $\mathcal{P}$ is a simple polygon (i.e., it has no self-intersections or holes), the obstruction to visibility must be caused by some part of the boundary of $\mathcal{P}$. Consider the collection of points $D \in \partial\mathcal{P}$ that obstruct the visibility from A to C; thus, $\overline{AC} \not\subset \mathcal{P}$. Among these, let $D^* \in \partial\mathcal{P}$ be a point such that the angle $\angle AD^*C$ is maximized. Since $\overline{AC} \not\subset \mathcal{P}$, this angle must be reflex, i.e., $|\angle AD^*C| > 180°$. This implies that D^* is a vertex of $\mathcal{P}$ with a reflex angle. Let $\overleftrightarrow{AD^*}$ denote the line passing through points A and D^*. Furthermore, since $C^* \in \overleftrightarrow{AD^*} \cap \partial\mathcal{P}$ is visible from A (if not, angle $\angle AD^*C$ cannot be maximized), so $\overleftrightarrow{AC^*} \subset \mathcal{P}$ and because $\overline{AD^*} \subset \overline{CD^*}$, the segment $\overline{AD^*}$ lies within $\mathcal{P}$, confirming that A is visible by D^*.

In both cases, we have shown that point A is visible by at least one reflex vertex of $\mathcal{P}$. Since $A \in \mathcal{P}$ was arbitrary, the result holds for all interior and boundary points of $\mathcal{P}$. □

Theorem 2 (Connectivity of reflex vertices)**.** *Let $\mathcal{P}$ be a simple polygon with a finite number of vertices, and suppose that $\mathcal{P}$ contains at least one reflex vertex. Then all reflex vertices of $\mathcal{P}$ can be connected by a polyline lying entirely within $\mathcal{P}$.*

Proof. Let $\mathcal{R} = \{R_1, R_2, \ldots, R_k\}$ denote the set of all reflex vertices of $\mathcal{P}$, where $k \geq 1$ and each $R_i \in \partial\mathcal{P}$ is a vertex, i.e. a point of polygon boundary $\partial\mathcal{P}$, at which the internal angle exceeds $180°$.

Case 1: If $k = 1$, there is only one reflex vertex. It is trivially connected to itself, so the statement holds.

Case 2: Let $k \geq 2$. We aim to construct a polyline connecting all reflex vertices using segments that lie entirely within $\mathcal{P}$. Select any two distinct reflex vertices, say A and B, where $A \in \mathcal{R}$, $B \in \mathcal{R}$.

Subcase 2a: If the segment $\overline{AB} \subset \mathcal{P}$, then A and B are directly connected.

Subcase 2b: If $\overline{AB} \not\subset \mathcal{P}$, the segment is obstructed by the boundary of $\mathcal{P}$. Since $\mathcal{P}$ is a simple polygon, the obstruction must be due to other parts of the boundary, not self-intersections or holes. Following a similar argument as used in reflex vertex visibility, we consider points along the boundary that block the visibility from A to B. Among those, there exists a vertex $D \in \partial\mathcal{P}$ such that the angle $\angle ADB$ is maximized. Because the segment $\overline{AB}$ is not entirely in $\mathcal{P}$, this angle must be greater than $180°$, implying that D is a reflex vertex. Clearly, the segment $\overline{AD} \subset \mathcal{P}$, so A and D are connected. If $\overline{DB} \subset \mathcal{P}$, then D and B are connected as well, and hence A, D, and B are connected via a polyline. If $\overline{DB} \not\subset \mathcal{P}$, we repeat the same argument: select a reflex vertex $D' \in \partial\mathcal{P}$ such that $\angle DD'B$ is maximized. Again, D' is a reflex vertex, and we connect D to D' via a segment in $\mathcal{P}$. If needed, this process is repeated. Since $\mathcal{P}$ has a finite number of vertices, and each iteration introduces a new reflex vertex not previously used, this process must terminate after a finite number of steps. Eventually, a polyline connecting A to B through intermediate reflex vertices is constructed.

By repeating this process for all pairs of reflex vertices, a single polyline connecting the entire set $\mathcal{R}$ within $\mathcal{P}$ can be constructed.

Hence, all reflex vertices in $\mathcal{P}$ can be connected by a polyline lying entirely within the polygon. □

III. Upper bounds on the shortest watchman route length based on the number of reflex vertices

In this section, we study upper bounds on the length of the shortest watchman route within a simple polygon, based on the number of reflex vertices it contains. We begin by formally defining a watchman route as a path within the polygon from which every point is visible from at least one position along the path.

We then analyze several cases: polygons with no reflex vertices, one reflex vertex, and two reflex vertices, showing how these configurations lead to increasingly tighter bounds on the route length. For polygons with three or more reflex vertices, we demonstrate that the general upper bound – equal to the perimeter of the polygon – cannot be improved in all cases. This is supported by a specific construction in which the shortest watchman route is (almost) necessarily equal to the perimeter. However, we also introduce a geometric technique we call *successive dual-edge triangulation*, and show that if such a triangulation is possible for the given polygon, the upper bound on the route length can be improved significantly – specifically, to below half the perimeter.

Definition 2 (Watchman route)**.** *Let $\mathcal{P}$ be a simple polygon. A* watchman route *is a continuous path $\Gamma \subseteq \mathcal{P}$ such that for every point $A \in \mathcal{P}$, there exists a point $B \in \Gamma$ for which $\overline{AB} \subseteq \mathcal{P}$. In other words, every point of the polygon is visible from at least one point along the route.*

For a polygon with no reflex vertices or with only a single reflex vertex, the shortest watchman route is straightforward to determine, and its length admits a tight and easily computable upper bound.

Theorem 3 (Upper bound on watchman route length in polygon with no reflex vertex)**.** *Let $\mathcal{P}$ be a simple polygon with no reflex vertices. Then the shortest watchman route in $\mathcal{P}$ has length zero, and any single point within $\mathcal{P}$ (including boundary points) serves as a valid watchman route.*

Proof. Since $\mathcal{P}$ has no reflex vertices, it is convex by definition. By Lemma 2, every point in $\mathcal{P}$ is visible from every other point. Therefore, any single point $A \in \mathcal{P}$ satisfies the condition that every point in the polygon is visible from A. Hence, the constant path $\Gamma = \{A\}$ is a valid watchman route. Because this path consists of a single stationary point, its length is zero, $|\Gamma| = |\{A\}| = 0$. Thus, the upper bound on the shortest watchman route length is exactly zero, and any point in $\mathcal{P}$ realizes this bound. □

Theorem 4 (Upper bound on watchman route length in polygon with one reflex vertex)**.** *Let $\mathcal{P}$ be a simple polygon with exactly one reflex vertex R. Then the shortest watchman route in $\mathcal{P}$ has length zero, and the reflex vertex R itself serves as a valid watchman route, so $\Gamma = \{R\}$.*

Proof. By Theorem 1, every point in $\mathcal{P}$ (interior or boundary) is visible from at least one reflex vertex. Since $\mathcal{P}$ contains exactly one reflex vertex R, it follows that every point in $\mathcal{P}$

is visible from R. Thus, the constant path $\Gamma = \{R\}$ is a valid watchman route. Since it consists of a single stationary point, its length is zero, $|\Gamma| = |\{R\}| = 0$. Therefore, the upper bound on the shortest watchman route length is zero, and R realizes this bound. $\square$

Remark 1. *While the reflex vertex R is guaranteed to serve as a valid watchman route, other points in $\mathcal{P}$ may also provide full visibility. However, R is a point that ensures this property without requiring further geometric analysis.*

We now consider the case of a simple polygon containing exactly two reflex vertices. In this setting, we show that a valid watchman route not only exists but can also be constructed with a length strictly less than half the perimeter of the polygon.

Theorem 5 (Upper bound on watchman route length in polygon with two reflex vertices)**.** *Let $\mathcal{P}$ be a simple polygon with exactly two reflex vertices, denoted R_1 and R_2. Then the shortest watchman route in $\mathcal{P}$ has length strictly less than half the perimeter of $\mathcal{P}$.*

Proof. By Theorem 1, every point in $\mathcal{P}$ is visible from at least one of the two reflex vertices R_1 or R_2. By Theorem 2, the two reflex vertices can be connected by a polyline lying entirely within $\mathcal{P}$. Since there are only two reflex vertices, the polyline reduces to the segment $\overline{R_1R_2}$. Therefore, any point visible from R_1 or R_2 is also visible from any point on the segment $\overline{R_1R_2}$, as visibility is preserved along the connecting segment. We now prove that the length of this segment is strictly less than half the perimeter of $\mathcal{P}$. Let the perimeter of $\mathcal{P}$ be denoted by $p = \mathrm{perim}(\mathcal{P})$. Traversing from R_1 to R_2 along the boundary of $\mathcal{P}$, there are two distinct polygonal paths: one in the clockwise direction, denoted Π_1, and one in the counter-clockwise direction, denoted Π_2. Let ℓ_1 and ℓ_2 be the respective lengths of these two paths. Then,

$$|\Pi_1| + |\Pi_2| = \ell_1 + \ell_2 = p = \mathrm{perim}(\mathcal{P}). \tag{1}$$

Since $\mathcal{P}$ is simple and R_1, R_2 lie on its boundary, the segment $\overline{R_1R_2}$ is a chord that spans across the interior of the polygon and lies entirely within $\mathcal{P}$. At least one of the polygonal (perimeter) paths Π_1 or Π_2 must contain at least one additional vertex, as a polygon $\mathcal{P}$ cannot consist of only two vertices. Without loss of generality, assume that the path Π_1 contains additional (distinct) vertices $A_1, A_2, \ldots, A_m$ for $m \geq 1$. By the triangle inequality, we have

$$\begin{array}{rcccl} |\overline{R_1R_2}| & < & |\overline{R_1A_1}| & + & |\overline{R_2A_1}|, \\ |\overline{R_2A_1}| & < & |\overline{A_1A_2}| & + & |\overline{R_2A_2}|, \\ \cdots & < & \cdots & + & \cdots, \\ |\overline{R_2A_{m-1}}| & < & |\overline{A_{m-1}A_m}| & + & |\overline{R_2A_m}|. \end{array}$$

Repeatedly substituting $|\overline{R_2A_i}|$ for $i \in \{1, 2, \ldots, m-1\}$ in each inequality with the corresponding bound from the next inequality yields

$$|\overline{R_1R_2}| < |\overline{R_1A_1}| + \sum_{i=1}^{m-1} |\overline{A_iA_{i+1}}| + |\overline{R_2A_m}| = |\Pi_1| = \ell_1.$$

The path Π_2 may contain no additional vertices, so it follows directly that $|\overline{R_1R_2}| \leq |\Pi_2| = \ell_2$. Hence, we have

$$|\overline{R_1R_2}| < \ell_1 \qquad \text{and} \qquad |\overline{R_1R_2}| \leq \ell_2,$$

and thus, $|\overline{R_1R_2}| + |\overline{R_1R_2}| = 2 \cdot |\overline{R_1R_2}| < \ell_1 + \ell_2$, so

$$|\overline{R_1R_2}| < \frac{\ell_1 + \ell_2}{2} \stackrel{(1)}{=} \frac{p}{2}.$$

So, the segment $\overline{R_1R_2}$ has length strictly less than half the perimeter, and it serves as a valid watchman route since every point in $\mathcal{P}$ is visible from R_1 or R_2 and thus from a point on $\overline{R_1R_2}$.

This establishes that the shortest watchman route in a polygon with two reflex vertices has length strictly less than half the polygon perimeter. $\square$

Remark 2. *The following examples illustrate the variability in shortest watchman routes for polygons with two reflex vertices:*

(a) In a butterfly-shaped polygon, the reflex vertices form symmetrical "wings", and a centrally placed point can see the entire region. Here, the shortest watchman route Γ has length zero, consistent with Theorem 5. See Fig. 1, part (a).

(b) In contrast, a polygon with a long rectangular base and two small inward-curved reflex "horns" may require a segment connecting the reflex vertices for full visibility. In this case, Γ approaches half the perimeter, showing that the bound in Theorem 5 is tight. See Fig. 1, part (b).

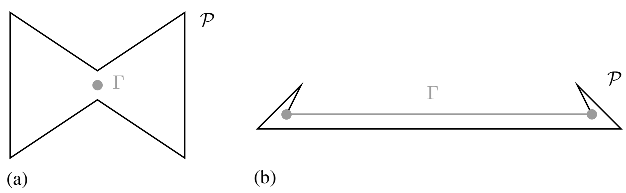

Fig. 1. Examples of polygons with two reflex vertices: (a) a butterfly-shaped polygon $\mathcal{P}$, where a single-point watchman route Γ (in grey) suffices; (b) a skinny polygon $\mathcal{P}$ with inward reflex "horns", where the shortest route Γ (in grey) approaches half the perimeter.

We now turn to the case of polygons with three or more reflex vertices. In general, the shortest watchman route in such polygons can approach the full perimeter, and we show that this upper bound cannot be improved in all cases. However, for certain polygonal structures, the application of a *successive dual-edge triangulation* – when possible – enables a significant improvement, reducing the upper bound on the route length to below half the perimeter.

Theorem 6 (Upper bound on watchman route length in polygon with three or more reflex vertices)**.** *Let $\mathcal{P}$ be a simple polygon with at least three reflex vertices. Then the length of the shortest watchman route in $\mathcal{P}$ is at most equal to* *the perimeter of $\mathcal{P}$. Moreover, this upper bound cannot be generally improved.*

Proof. By Lemma 1, every point in $\mathcal{P}$ is visible from at least one point on the boundary $\partial\mathcal{P}$, so a route along the boundary guarantees full visibility. Thus, the perimeter is a valid upper bound for the shortest watchman route. To potentially improve this bound, one might consider a polyline connecting all reflex vertices, since Theorem 1 ensures that every point in $\mathcal{P}$ is visible from at least one of them. However, this polyline may still approach the perimeter length. To show this, we construct a family of polygons, see Fig. 2, with alternating convex and reflex vertices arranged as repeated triples (A_i, B_i, R_i), where R_i is reflex. Each "tooth" – a subregion formed near vertex B_i, which lies behind a reflex corner – necessitates that the watchman visit the corresponding reflex vertex R_i to maintain visibility, particularly of the B_i vertices. Connecting the reflex vertices yields a route Γ whose length closely matches the perimeter, as the segments $\overline{R_i R_{i+1}}$ are only slightly shorter than the corresponding polygonal arcs $\overline{R_i A_{i+1} B_{i+1} R_{i+1}}$. Therefore, while the perimeter is always a valid upper bound, it is asymptotically tight in general and cannot be improved without additional structural assumptions. $\square$

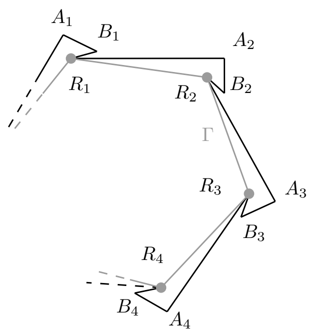

Fig. 2. A polygon constructed from repeated vertex triples (A_i, B_i, R_i), where A_i and B_i are convex and R_i is reflex. The visibility constraints force the watchman to visit each reflex vertex, resulting in a route Γ (in grey) whose length approaches the full perimeter. This demonstrates that the general upper bound cannot be improved for polygons with three or more reflex vertices.

Definition 3 (Successive dual-edge triangulation)**.** *Let $\mathcal{P}$ be a simple n-sided polygon with vertices $A_1, A_2, \ldots, A_n$. A triangulation $\mathcal{T} = \{T_1, T_2, \ldots, T_m\}$ is called a* successive dual-edge triangulation *of polygon $\mathcal{P}$ if the following conditions are satisfied,*

- *each triangle $\tau_j \in \mathcal{T}$ is formed by three vertices of $\mathcal{P}$ and lies entirely within $\mathcal{P}$;*
- *the union of all triangles in $\mathcal{T}$ exactly covers the interior of $\mathcal{P}$ without overlaps or gaps;*
- *the first triangle $\tau_1 = \triangle A_{i_1} A_{i_2} A_{i_3}$ shares at least two of its edges with the boundary $\partial\mathcal{P}$ of polygon $\mathcal{P}$;*
- *Every subsequent triangle τ_j for $j > 1$ shares one edge with the previously constructed triangle τ_{j-1}, and one or two of its edges must coincide with edges of $\partial\mathcal{P}$. Consequently, triangle τ_j has two edges not shared with $\partial\mathcal{P}$ for $j < m$, and one such edge if $j = m$.*

Fig. 3 illustrates an example of a polygon with a successive dual-edge triangulation. The triangulation proceeds from triangle $\tau_1 = \triangle A_1 A_2 A_3$ and continues through adjacent triangles such as $\tau_2 = \triangle A_1 A_3 A_{24}$ and $\tau_3 = \triangle A_3 A_{23} A_{24}$, ultimately reaching $\tau_{21} = \triangle A_{12} A_{13} A_{14}$. The dotted lines indicate the internal diagonals added during the successive construction.

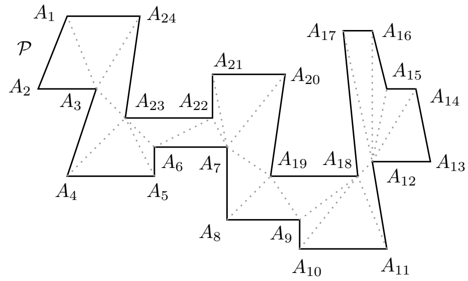

Fig. 3. An example of a polygon $\mathcal{P}$ with 24 vertices, demonstrating a successive dual-edge triangulation. Dotted lines represent internal diagonals added in a step-by-step manner. The triangulation includes, among others, the triangles $\tau_1 = \triangle A_1 A_2 A_3$, $\tau_2 = \triangle A_1 A_3 A_{24}$, $\tau_3 = \triangle A_3 A_{23} A_{24}$, and continues up to $\tau_{21} = \triangle A_{12} A_{13} A_{14}$.

Theorem 7 (Upper bound improvement via successive dual-edge triangulation)**.** *Let $\mathcal{P}$ be a simple polygon with perimeter p, and suppose that $\mathcal{P}$ admits a successive dual-edge triangulation as defined in Definition 3. Then the length of the shortest watchman route in $\mathcal{P}$ is strictly less than $\frac{p}{2}$.*

Proof. Let $\mathcal{T} = \{\tau_1, \tau_2, \ldots, \tau_m\}$ be a successive dual-edge triangulation of $\mathcal{P}$, as defined in Definition 3. We construct a watchman route Γ by connecting the midpoints of edges not shared with the polygon boundary $\partial\mathcal{P}$ that appear in the triangles of $\mathcal{T}$.

For the initial triangle τ_1, we add a segment connecting the midpoint of one boundary edge to the midpoint of the edge not shared with $\partial\mathcal{P}$. For each subsequent triangle τ_j with $1 < j < m$, we add a segment connecting the midpoints of its two edges that are not shared with the boundary. For the final triangle τ_m, we again add a segment connecting the midpoint of the edge not shared with $\partial\mathcal{P}$ to the midpoint of one boundary edge. By the construction of the triangulation, these segments are connected consecutively and lie entirely within the interiors of the corresponding triangles. Since the union of all triangles in $\mathcal{T}$ covers the entire polygon $\mathcal{P}$, the resulting path Γ guarantees visibility of the entire region.

Each segment in the route corresponds to half the length of a polygon boundary edge (or two such edges in the case of τ_1 and τ_m). Since no boundary edge is used in more than one triangle, the total contribution to the route is at most the sum of half-lengths of all boundary edges. Moreover, the first and last triangles each involve two boundary edges but contribute only one midline segment, meaning that two boundary edges are not halved and do not contribute to the route.

Therefore, the total length of the constructed watchman route Γ is strictly less than half the perimeter, $|\Gamma| < \frac{p}{2}$. $\square$

Figure 4 illustrates the construction of a watchman route based on a successive dual-edge triangulation of a 24-vertex polygon, using midlines of interior triangles to ensure visibility.

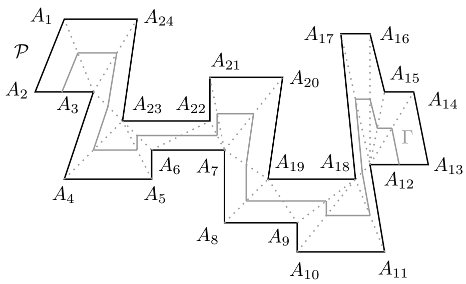

Fig. 4. A simple polygon $\mathcal{P}$ with 24 vertices, along with its successive dual-edge triangulation (indicated by dotted lines) and a watchman route Γ (shown as a bold grey polyline) constructed by connecting midpoints of selected triangle edges. The route lies entirely within the polygon and ensures full visibility while having total length strictly less than half the perimeter.

The example in Fig. 4 also illustrates the practical applicability of our approach. Although the polygon has a complex structure with 24 vertices and numerous reflex angles – resembling environments such as industrial halls or an irregular courtyard of a large private property – it admits a successive dual-edge triangulation. By connecting midpoints of interior triangle edges, we construct a watchman route that ensures full visibility and is provably shorter than half the perimeter. This method serves not only as a theoretical upper bound but also as a constructive algorithm for efficient route planning in scenarios where full geometric detail is available. Compared to strategies that rely on exhaustive perimeter traversal, our approach can substantially reduce route length. However, as visible in the figure, further local improvements may be possible in subregions without reflex vertices, suggesting opportunities for route refinement beyond the triangulation-based construction.

IV. Conclusion

We have analyzed upper bounds on the length of the shortest watchman route in simple polygons under varying structural conditions. For polygons with no reflex vertex, the upper bound is trivial: any point within the polygon suffices, and the shortest watchman route has length zero. Similarly, in polygons with a single reflex vertex, the upper bound remains zero, with the reflex vertex itself serving as a valid watchman position. In polygons with exactly two reflex vertices, we showed that the shortest watchman route can be constructed to have a length strictly less than half the polygon's perimeter, and we further demonstrated that this bound cannot be improved in general. For polygons with three or more reflex vertices, the general upper bound equals the full perimeter, and we likewise showed that this bound is tight in the worst case. However, if the polygon admits a successive dual-edge triangulation, the upper bound can be reduced to less than half the perimeter.

This triangulation not only offers a sharper theoretical bound but also enables a constructive algorithm when the full geometric structure of the polygon is known. It provides a systematic way to plan efficient watchman routes in practical scenarios. Nonetheless, we note that the triangulation-based route may not always be optimal, and local refinements – particularly in convex or reflex-free subregions – could lead to even shorter paths.

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