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A New Approach to the Beam Calculus Made of Functionally Graded Materials

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***Abstract*—The use of functionally graded materials is increasingly used, from the simplest structural elements to the most complex or dedicated ones, in more and more fields of engineering creation. Under these conditions, both the manufacture of these materials and the development of their calculation are also in an appropriate development. This work is placed in the field of developing the calculation of mechanical structures from functionally graded materials, in conditions of accuracy, efficiency, economy and accessibility for designers and scientific researchers in the field. The beam we researched is the Euler-Bernoulli beam - the type most approached in mechanical engineering constructions. The new approach proposed by the authors is based on the use of two concepts (defined and used) original for the field addressed: the concept of multilayer material and the concept of equivalent material. In this work, the research results refer only to the calculation of displacements in bars subjected to bending. In the research carried out, the variation (according to the material law) of both Young's modulus and Poisson's ratio was taken into account. The method, methodology and models presented are theoretically substantiated and supported by practical examples, with general validity. The validation of the new approach in the calculation of bent beams is carried out on simple examples, with well-known analytical solutions, by applying both to the beam made of homogeneous and isotropic material, and to the beam made of a functionally graded material, made according to the power law of the material, for different values of the power coefficient. An important aspect of the authors' research is the fact that the approach to the calculation of bent bars made of functionally graded materials is accessible to numerical calculation without developing an appropriate software, but using general calculation programs using the finite element method. Based on our research, we state, without demonstrating in this paper, that the proposed approach is also suitable for calculation by numerical methods of the meshless and meshfree type. The work is distinguished by its novelty and originality and is part of the general effort to develop the calculation of structures made of functionally graded materials.**

***Index Terms*—Functionally Graded Materials; finite element method; beam calculus.**

I. Introduction

THIS paper is the result of my own scientific research in the field of calculation of structures from functionally graded materials. The structural element presented in this paper is a straight beam, of the Euler-Bernoulli type, loaded in bending.

The beam is a very widespread structural element in engineering constructions and also a structural element very suitable to be built from functionally graded materials [1], [2], [3]. The calculation of structures from functionally graded materials represents a problem with great mathematical difficulties stemming from the variation according to a certain law of its properties. There are several material laws [4], [5], the most used, to which our work also refers, is the power law. This work proposes to solve the calculation of the displacements of bent bars in an original, accessible and efficient way by introducing the concepts of multilayer material and equivalent material. The two concepts can be used independently or in combination. The models, methodology and examples that are presented make the calculation of functionally graded bars (FGB) an accessible problem for both analytical and numerical calculation, such as the finite element method but also other methods such as the Galerkin Free Element Method, the Smoothed Particle Hydrodynamics Method, etc. can be used. The numerical solutions in this paper are carried out with the finite element method - the most well-known and used method - and the use of the Ansys program (the highly reliable software product), so that accessibility is as high as possible, using common programs. The illustrative examples are based on the well-known case for the calculation of bent straight bars made of homogeneous and isotropic materials, namely the bar simply supported at the ends, loaded with a uniformly distributed load. Considering the possibilities offered by the use of the two concepts (multilayer material and equivalent material), our research also considered the evaluation of taking into account the influence of the Poisson's ratio, as a constant value or with a variable value according to the material law. The concepts and methodology presented in this paper are of a general nature, which allows those interested to develop research for other structural elements. The authors' achievements include published works on the calculation of other structures, such as plates, thick-walled tubes, etc. The introduction of the two concepts of approach to calculus, through their originality and especially through the proposed methodology, represents the original and novel part, which makes the calculation of functionally graded bars an accessible and efficient calculation, with current means within the reach of any researcher.

II. Research Object

Our research focuses on a straight beam made of a functionally graded material according to the power law, based on two materials: alumina and aluminum. Their properties are presented in Table I.

Table I. Material Properties

Materials	Ceramic material (Al_2O_3)	Aluminum (Al)
Position	Top	Bottom
E [Pa]	$3.8 \cdot 10^{11}$	$7 \cdot 10^{10}$
$\nu[-]$	0.22	0.33
$\rho[kg/m^3]$	3960	2700

The functionally analyzed graded beam has the characteristics presented in Figure 1. The power law [3] is represented by the relation (1), with reference to Young's modulus (E). Analogously, the relation is written with reference to Poisson's ratio (ν). Analogously, the power law is written for any other properties, for which the values on the extreme faces are known.

$$E(y) = E_b + (E_t - E_b)\left(0.5 + \frac{y}{h}\right)^k \tag{1}$$

The indices "t" and "b" signify the "top" and "bottom" positions, respectively, in relation to the beam – Figure 1. The parameter "k" is called the power coefficient, which theoretically can have values from zero to infinity.

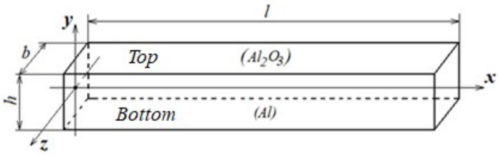

Fig. 1. Functionally graded beam

It is also found that small values of the power coefficient ($k<1$) lead to an increased influence of the material with superior characteristics (the ceramic material in this case).

For the value one of the power coefficient, the variation of all properties is linear. Figures 2 and 3 also show us a different curvature of the material properties variation curves, for the power coefficient value ranges: $k<1$, respectively $k>1$.

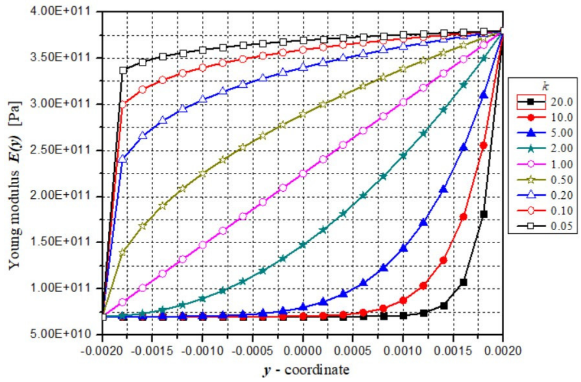

Fig. 2. Variation of Young's modulus over the height of the section

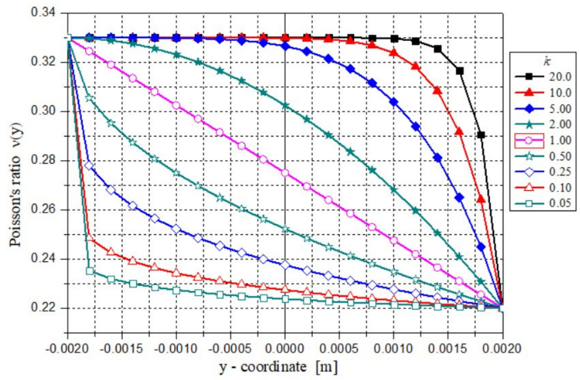

Fig. 3. Variation of Poisson's ratio over the height of the section

The dimensions of the analyzed beam have the values: $l=0.20\ m$ si $b=h=0.004\ m$ The distributed load is $p=1\cdot10^5\ N/m$.

III. The Concept of Multilayer Material

The concept of multilayer material considers that the functionally graded material is composed of a number of layers of different materials, with homogeneous and isotropic properties (Figure 4) [6]. Each layer (j) is assigned the material properties calculated with the relation (1).

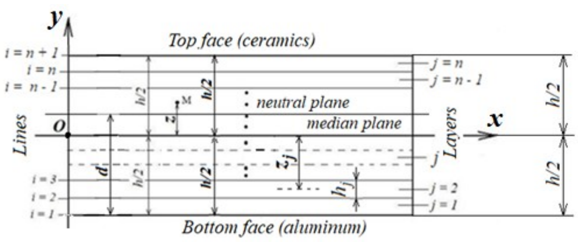

Fig. 4. Illustration of the multilayer material concept

For an infinite number of layers, the variation of the properties becomes continuous, according to the adopted law. In practice, one must work with a finite number of layers.

This concept causes the continuous variation of the properties to be replaced by a step variation, as exemplified in Figure 5, regarding the variation of the Young's modulus.

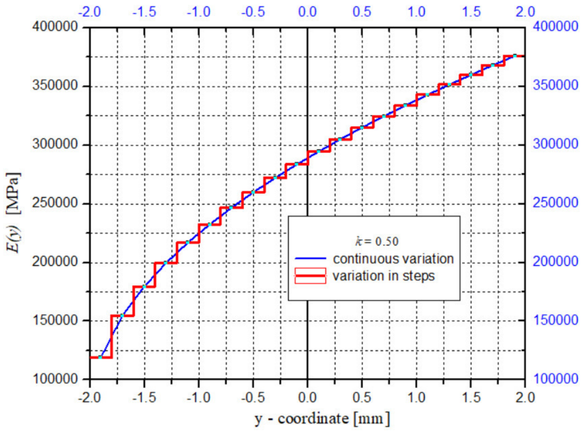

Fig. 5. Stepwise variation of Young's modulus

TABLE II. THE LAYERS VALUES OF ELASTIC CONSTANTS

Layers	E(y)	ρ(y)	ν(y)
1	3.278E+11	3747.77	0.311
2	3.423E+11	3806.94	0.317
3	3.494E+11	3835.58	0.319
4	3.541E+11	3854.84	0.321
5	3.577E+11	3869.44	0.322
6	3.606E+11	3881.24	0.323
7	3.631E+11	3891.15	0.324
8	3.652E+11	3899.70	0.325
9	3.670E+11	3907.23	0.325
10	3.687E+11	3913.96	0.326
11	3.702E+11	3920.05	0.327
12	3.715E+11	3925.61	0.327
13	3.728E+11	3930.73	0.327
14	3.740E+11	3935.48	0.328
15	3.751E+11	3939.90	0.328
16	3.761E+11	3944.04	0.329
17	3.770E+11	3947.94	0.329
18	3.779E+11	3951.62	0.329
19	3.788E+11	3955.10	0.330
20	3.796E+11	3958.41	0.330
Average values	3.654E+11	3900.84	0.325

The property values assigned to each layer (homogeneous and isotropic) are presented in Table II. These values are calculated for the power coefficient $k = 0.05$.

$$t_{\max} \le \frac{h}{20}\,[m] \tag{2}$$

The number of layers is adopted based on one's own experience, by calculating the maximum thickness of the layers. Using relation (2), it resulted $t_{\max} = 0.0002$ m.

A. Calculus of bending stiffness

From the study of straight bars subjected to bending, the relationships (2) and (3) are known [7], [8]:

$$u = -z\frac{\mathrm{d}w}{\mathrm{d}x} \tag{2}$$

$$\varepsilon_x = \frac{\mathrm{d}u}{\mathrm{d}x} = -y\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} \tag{3}$$

Since the stress state in the beam is linear (monoaxial), for each layer k the following relationship can be written [9]:

$$(\sigma_x)_k = E_k \cdot (\varepsilon_x)_k = -y \cdot E_k \frac{\mathrm{d}^2 v}{\mathrm{d}x^2} \tag{4}$$

In the cross section of the multilayer beam, the relationship between bending moment and stresses is written [7]:

$$M_z = \sum_k \int_{y_{k-1}}^{y_k} (\sigma_x)_k \cdot (y - y_c)\cdot dy \tag{5}$$

In relation (5), $y_c = d$ is the coordinate that defines the position of the neutral plane (Figure 4).

Its calculation is done [10] with the relation,

$$y_c = \frac{\sum_k S_k}{\sum_k A_k} \tag{6}$$

representing the ratio between the static moment of the each layer area with respect to the lower side of the cross-section and the static moment of the entire section with respect to the same side. By introducing relation (6) into relation (5), after some mathematical operations, the relation is obtained:

$$M_z = -\frac{\sum_k E_k \cdot \frac{y_k^3 - y_{k-1}^3}{3} \cdot \sum_k E_k \cdot (y_k - y_{k-1}) - \left(\sum_k E_k \frac{y_k^2 - y_{k-1}^2}{2}\right)^2}{\sum_k E_k \cdot (y_k - y_{k-1})} \cdot \frac{\mathrm{d}^2 v}{\mathrm{d}x^2} \tag{7}$$

Comparing relation (7) with the differential equation of the deformed average fiber, we arrive at the identification of the bending stiffness:

$$EI = K = \frac{A \cdot C - B^2}{A} \tag{8}$$

In relation (8) the following notations were made:

$$A = \sum_k E_k \cdot (y_k - y_{k-1}) \tag{9}$$

$$B = \sum_k E_k \frac{y_k^2 - y_{k-1}^2}{2} \tag{10}$$

$$C = \sum_k E_k \cdot \frac{y_k^3 - y_{k-1}^3}{3} \tag{11}$$

For the considered FGB and for the power coefficient $k = 0.05$, $A = 1.46178 \cdot 10^9$ N/m, $B = 2.97939 \cdot 10^6$ N and $C = 7997.95$ Nm, resulting in bending stiffness:

$$K = 1925.3795 \text{ Nm}^2 \tag{12}$$

Knowing the bending stiffness allows the calculation of the displacements of bent straight bars with the known relations for homogeneous and isotropic straight bars. The relations depend on the mode of support and are well known in the literature.

IV. The Concept of Equivalent Material

The concept of equivalent material allows the replacement of a functionally graded material (multilayer) with a fictitious, homogeneous and isotropic one, which has the same behavior as the real material. For the calculation of the displacements of bent straight bars, the equivalent material must have the same bending stiffness. For the calculation of the equivalent bending stiffness $(EI)_{ech} = K_{ech}$, we will define the bending stiffness by analogy with the multilayer plate [7].

Thus,

- for a homogeneous and isotropic plate,

$$D = \frac{Eh^3}{12(1-\nu^2)} = \frac{E \cdot I}{1-\nu^2} \tag{13}$$

- for the multilayer plate, we can write,

$$D^* = \frac{E_t I^*}{1-\nu^2} \tag{14}$$

where the parameter I^* (axial moment of inertia for width equal to unity) is defined by the relation,

$$I^* = \int_{-h/2}^{h/2} \frac{E_k}{E_t} y_k^2 \, dy \tag{15}$$

where y_k represents the distance of the average fiber of the layer k from the neutral axis, and the ratio E_k / E_t is a constant c_k with different values from one layer to another. In relation (14), we recognize the calculus relation of the axial inertia moment of each laye area, with the equivalent width of each layer equal to c_k, from the neutral axis. In the case of bars, being a monoaxial state of tension, the denominator $(1-\nu^2)$ is replaced by the value one. Therefore, for the calculation of the bending stiffness of functionally graded straight bars we can write:

$$D^* = E_t I^* = K_{ech} \tag{16}$$

$$I^* = \int_{-h/2}^{h/2} c_k \cdot y_k^2 \cdot dy \tag{17}$$

$$c_k = \frac{E_k}{E_t} \tag{18}$$

After performing the integrals, for n layers we can write:

$$I^* = \left[\frac{c_1 h_1^3}{12} + c_1 h_1 d_{G1}^2\right] + \left[\frac{c_2 \cdot h_2^3}{12} + c_2 h_2 d_{G2}^2\right] + \ldots + \left[\frac{c_n h_n^3}{12} + c_n h_n d_{Gn}^2\right] \tag{19}$$

For the considered FGB, for the power coefficient $k = 0.05$, the calculus of the equivalent inertia moment leads to the value $I^* = 5.07240 \cdot 10^{-9}$ m^4.

Using the relation $D^* = E_t I^* = K_{ech}$, the equivalent bending stiffness is:

$$K_{ech} = 1925.3804 \text{ Nm}^2 \tag{20}$$

The values of bending stiffness found for those two concepts are practically identical (error of 0.00013%). The two methods, concepts are therefore concordant.

V. Numerical approach

The numerical analysis is performed with the finite element method [13], [14], [15], based on the concept of multilayer material, using data of Table II. The finite element used is SOLID185 from the Ansys program library [16], [17].

This finite element (8-node brick) is used in two variants: with the multilayer finite element option (Figure 6) and with the structural finite element option (regular) [11], [12]. The finite element model is performed considering the beam made up of 20 different layers, without sliping between them.

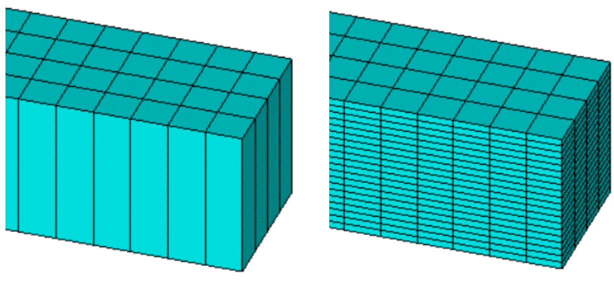

Fig. 6. Details of multilayer finite element modeling

The deformed state of the functionally graded bar shown in Figure 7, where the maximum displacement $v_{max} = 0.001083$ represents the case of the beam made of the material with the power coefficient $k = 0.05$ and Poisson's ratio $\nu = 0$.

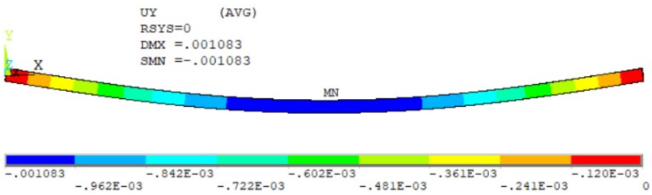

Fig. 7. Deformed state of the functionally graded beam

TABLE III. BENDING STIFFNESS

k	A	B	C	$EI = K$
[-]	$[\mathrm{N/m}] \cdot 10^9$	[N]	[Nm]	$[\mathrm{Nm}^2]$
0.05	1.46178	2979385.67	7997.95	1925.3779
0.10	1.40854	2921669.24	7892.75	1832.4824
0.25	1.27346	2763964.10	7596.60	1597.6017
0.50	1.10745	2543269.04	7159.49	1318.8262
1.00	0.90000	2212300.00	6449.20	1011.1208
2.00	0.69307	1798451.83	5454.45	787.6744
4.00	0.52748	1384085.15	4316.06	684.3028
6.00	0.45637	1176390.67	3681.88	649.4752
8.00	0.41675	1051368.99	3276.87	624.4811

TABLE IV. EQUIVALENT BENDING STIFFNESS

Layer No.	E_k	y_{Gk}	$c_k = b$	I_k
[-]	[Pa] $\cdot 10^{11}$	[m]	[-]	$[\mathrm{m}^4] \cdot 10^{-10}$
1	3.27786	-0.0019	0.862594737	6.48663
2	3.42342	-0.0017	0.900900000	5.44986
3	3.49388	-0.0015	0.919442105	4.35704
4	3.54128	-0.0013	0.931915789	3.34392
5	3.57721	-0.0011	0.941371053	2.44537
6	3.60622	-0.0009	0.949005263	1.67699
7	3.63060	-0.0007	0.955421053	1.04766
8	3.65164	-0.0005	0.960957895	5.63107
9	3.67017	-0.0003	0.965834211	2.27382
10	3.68673	-0.0001	0.970192105	4.35278
11	3.70172	0.0001	0.974136842	1.39352
12	3.71540	0.0003	0.977736842	1.40545
13	3.72800	0.0005	0.981052632	4.24977
14	3.73967	0.0007	0.984123684	8.68612
15	3.75055	0.0009	0.986986842	1.47265
16	3.76074	0.0011	0.989668421	2.23814
17	3.77033	0.0013	0.992192105	3.16603
18	3.77937	0.0015	0.994571053	4.25715
19	3.78794	0.0017	0.996826316	5.51228
20	3.79608	0.0019	0.998968421	6.93211
			$I_{ech} = \sum_k I_k =$	50.6679
			$D^* = E_t I^* = K_{ech} =$	1925.3804

TABLE V. COMPARATIVE EVALUATION OF STIFFNESS

k	$EI = K$	$K_{ech}(k)$	E_{rr}.
[-]	$[\mathrm{Nm}^2]$	$[\mathrm{Nm}^2]$	[%]
0.05	1925.3779	1925.3804	-0.00013
0.10	1832.4824	1832.4822	0.00001
0.25	1597.6017	1597.6045	-0.00018
0.50	1318.8262	1318.8237	0.00019
1.00	1011.1208	1011.1208	0.00000
2.00	787.6744	787.6768	-0.00030
4.00	684.3028	684.3033	-0.00007
6.00	649.4752	649.4762	-0.00015
8.00	624.4811	624.4782	0.00046

VI. COMPARATIVE RESULTS

For a series of values of the power coefficient k, Table III presents the values of some intermediate parameters and the value of the bending stiffness $K = EI$ of the beam, according to the concept of multilayer material. Table IV presents the values of some parameters and of equivalent bending stiffness $K_{ech} = (EI)_{ech}$ in accordance with the concept of equivalent material, all of this for the power coefficient $k = 0.05$.

Repeating the calculations in Table IV for the other values of the power coefficient, we find the values in Table V, comparatively with the stiffness values in Table III.

Rezultatele calculului analitic si numeric sunt prezentate sintetic, comparativ, in Tabelul VI, pentru mai multe valori ale coeficientului putere, pentru diferite valori ale coeficientului Poisson, pentru cazul barei simplu rezemate la capete, incarcata cu o sarcina uniform distribuita. Relatia analitica de calcul este:

$$v_{max} = \frac{5}{384} \cdot \frac{pl^4}{EI} = \frac{5}{384} \cdot \frac{pl^4}{K} \tag{21}$$

VII. CONCLUSIONS

The paper deals, in an original manner, with a topic of great relevance and importance for mechanical engineering – the calculation of beams made of functionally graded materials.

The originality and consequently the novelty brought by the work lies in the manner of solving, by using the concepts of equivalent material and multilayer material. These concepts can be used both independently and in combinations, leading to the greatest possible efficiency of the calculation.

The numerical exemplification of the use of the presented concepts, methods and methodologies validates their use for calculating the displacements of functionally graded beams.

The values in Table VI demonstrate through quantitative arguments that the influence of the Poisson's ratio is small, but it exists and can be evaluated.

TABLE VI. COMPARATIVE EVALUATION OF STIFFNESS

k	$v_{\max}^{analytical}$	$v_{\max}^{FEM}$	E_{rr}
[-]	[m]	[m]	[%]
$\nu=0$			
0.05	0.001082	0.001083	*0.19*
0.10	0.001137	0.001138	*-0.27*
0.25	0.001304	0.001305	*-0.46*
0.50	0.001580	0.001581	*-0.02*
1.00	0.002060	0.002062	*0.02*
2.00	0.002645	0.002648	*0.19*
4.00	0.003044	0.003048	*0.15*
6.00	0.003208	0.003211	*-0.07*
8.00	0.003336	0.003340	*-0.12*
		Average error =	***-0.04***
$\nu=0.30$			
0.05	0.001082	0.001084	*-0.18*
0.10	0.001137	0.001139	*-0.19*
0.25	0.001304	0.001306	*-0.15*
0.50	0.001580	0.001582	*-0.15*
1.00	0.002060	0.002063	*-0.13*
2.00	0.002645	0.002649	*-0.15*
4.00	0.003044	0.003050	*-0.18*
6.00	0.003208	0.003213	*-0.16*
8.00	0.003336	0.003342	*-0.18*
		Average error =	***-0.16***
$\nu=\nu(k)$			
0.05	0.001082	0.001084	*-0.18*
0.10	0.001137	0.001138	*-0.10*
0.25	0.001304	0.001305	*-0.07*
0.50	0.001580	0.001579	*0.04*
1.00	0.002060	0.002058	*0.12*
2.00	0.002645	0.002641	*0.15*
4.00	0.003044	0.003042	*0.08*
6.00	0.003208	0.003206	*0.05*
8.00	0.003336	0.003334	*0.06*
		Average error =	***0.02***

Although for the three cases of considering the Poisson's ratio the errors are below 1%, the case closest to the analytical calculation is the one in which the Poisson's ratio is considered in accordance with the law of variation – $\nu=\nu(k)$ – when the average error is the smallest, reaching the value of 0.02%.

In this paper we have only referred to the calculation of displacements, but the concepts, methods and methodologies presented are generally valid for the calculation of rotations or taking into account the effects of shear force. Also, our way of solving functionally graded beams is independent of the beam support mode.

This new approach to the calculation of bars made of functionally graded materials has been used for other structural elements as well.

The results of our research, although presented synthetically in this paper, highlight the validity, efficiency and accessibility of the new way of approaching the calculation of functionally graded beams

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