

# A triangulation-free guard bound for polygonal domains with holes: Structural guarding by outer reflex and hole vertices

Lubomír Štěpánek<sup>1, 2, 3</sup>

ORCID: 0000-0002-8308-4304

<sup>1</sup>Department of Statistics and Probability

<sup>2</sup>Department of Mathematics

Faculty of Informatics and Statistics

Prague University of Economics and Business

W. Churchill's square 4, 130 67 Prague, Czech Republic

lubomir.stepanek@vse.cz

&

<sup>3</sup>Institute of Biophysics and Informatics

First Faculty of Medicine

Charles University

Salmovská 1, Prague, Czech Republic

lubomir.stepanek@lf1.cuni.cz

**Abstract**—We study point guarding of polygonal domains with holes, a classical visibility problem in computational geometry. The goal is to place guards so that every point of the domain is visible from at least one guard. We consider a guard set defined directly by the boundary structure of the domain: all reflex vertices of the outer boundary together with all vertices of the holes.

We prove that this guard set always covers the entire domain, including the hole boundaries. The proof is based on Euclidean shortest paths and avoids triangulation. In this way, the result provides a simple structural guarding rule that differs from classical triangulation-based constructions.

Classical bounds for polygons with holes are usually stated in terms of the total number of vertices and the number of holes. Our bound is of a different type: it is parameterized by the number of outer-boundary reflex vertices and the total number of hole vertices, rather than relying on a triangulation-based construction. Although it is not uniformly smaller than the classical bounds, it can be substantially better for domains with few outer reflex vertices or with holes of low total boundary complexity. We also show that the dependence on the number of outer reflex vertices is already tight for suitable families of simple polygons.

## I. INTRODUCTION

**V**ISIBILITY-BASED covering problems in polygonal environments form a classical topic in computational geometry. In the standard art-gallery problem, one asks for a set of guards whose visibility regions cover the whole polygonal domain. For a simple polygon with  $n$  vertices, Chvátal proved that  $\lfloor \frac{n}{3} \rfloor$  guards are always sufficient [1], and Fisk later gave the well-known short proof based on triangulation and 3-coloring [2]. Classical background on these problems can

be found, for example, in O'Rourke's monograph [3] and in Shermer's survey [4].

For polygonal domains with holes, several upper bounds are stated in terms of the total number of vertices  $n$  and the number of holes  $h$ . In particular, a standard vertex-guard bound is  $\lfloor \frac{n+2h}{3} \rfloor$ , published by [3], while for point guards the sharper bound  $\lfloor \frac{n+h}{3} \rfloor$  is known, as comes from [5], [6]. A closely related classical conjecture, attributed to Shermer [4], states that  $\lfloor \frac{n+h}{3} \rfloor$  vertex guards should suffice for polygons with  $h$  holes; this is known to hold for  $h = 1$ , see [7].

Besides their theoretical interest, visibility models in polygons are also relevant in surveillance, sensor placement, inspection planning, and watchman-type route design in geometric environments. In such settings, structurally simple guard placements may be attractive even when they are not globally optimal.

In this paper we consider a different type of upper bound. Let  $P$  be a polygonal domain with holes. Denote by  $n_r$  the number of reflex vertices on the outer boundary, and by  $n_h$  the total number of vertices of all holes. We prove that placing guards at all reflex vertices of the outer boundary and at all vertices of the holes is sufficient to see every point of  $P$ , including the hole boundaries. Hence  $n_r + n_h$  guards always suffice. We also show that the dependence on  $n_r$  is not merely an artifact of the proof: for simple polygons, there exist infinite families for which the required number of guards is exactly  $n_r$ .

Our main point is not that this bound replaces the classical  $\{n, h\}$ -based bounds. In general, it does not. Rather, it provides a structurally simple and triangulation-free alternative, defined directly from the boundary geometry of the domain.

This is the main distinction from classical proofs and constructions based on triangulation. Moreover, in domains where the outer boundary has few reflex vertices, or where the holes have low total boundary complexity, the quantity  $n_r + n_h$  may be substantially smaller than bounds expressed only through  $n$  and  $h$ .

The present result also fits into a broader line of work on visibility-based geometric bounds. In earlier work, lower bounds were studied for the proportion of a polygon visible from a single point, first for general simple polygons [8] and later in an extended treatment including orthogonal polygons [9]. More recently, upper bounds on shortest watchman routes were analyzed using reflex-vertex structure and triangulation-based ideas [10]. The contribution of the present paper is different in scope: it concerns a static guard set for polygonal domains with holes and gives a shortest-path-based proof that avoids triangulation entirely.

Among works closest in spirit, we also note the result of Alipour on polygons with holes showing a  $\lfloor \frac{n+h}{3} \rfloor$ -type theorem for guarding the vertices of the polygon, together with coverage of the entire outer boundary [11]. Our contribution differs in that it guards the entire polygonal domain and is parameterized explicitly by the outer reflex structure and the total complexity of the holes, rather than only by the total number of vertices and holes.

The paper is organized as follows. After brief preliminaries, we state and prove the main theorem using shortest-path geometry. We then discuss how the bound  $n_r + n_h$  compares with classical art-gallery bounds and indicate classes of domains in which this parameterization is favorable.

## II. PROBLEM STATEMENT AND PRELIMINARIES

Let  $P$  be a polygonal domain in the plane. Its boundary consists of one outer boundary, which is a simple polygon, and  $h \geq 0$  pairwise disjoint hole boundaries, each of which is also a simple polygon contained in the interior of the outer boundary. We write  $n$  for the total number of vertices of all boundary components of  $P$ ,  $n_r$  for the number of reflex vertices of the outer boundary, and  $n_h$  for the total number of vertices of all holes.

Fig. 1 illustrates the notation used throughout the paper, in particular the parameters  $n$ ,  $h$ ,  $n_r$ , and  $n_h$ .

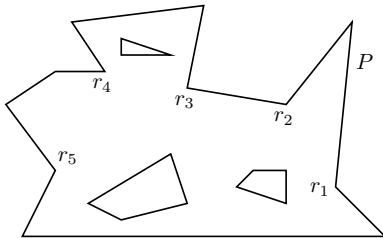


Fig. 1. A polygonal domain  $P$  with holes. The outer boundary has 12 vertices and the holes have  $n_h = 4 + 4 + 3 = 11$  vertices in total, so the total number of vertices of all boundary components is  $n = 12 + 11 = 23$ . The number of holes is  $h = 3$ , and the outer boundary has  $n_r = 5$  reflex vertices, namely  $r_1, r_2, \dots, r_5$ .

For points  $x, y \in P$ , we say that  $x$  sees  $y$  if the closed segment  $\overline{xy}$  is contained in  $P$ . A set  $G \subseteq P$  is a guard set for  $P$  if every point of  $P$  is visible from at least one point of  $G$ .

We consider the guard set

$$G := R_{\text{out}} \cup V_{\text{holes}},$$

where  $R_{\text{out}}$  is the set of all reflex vertices of the outer boundary and  $V_{\text{holes}}$  is the set of all vertices of all holes. Thus,

$$|G| = n_r + n_h.$$

For two points  $x, y \in P$ , we denote by  $\pi_P(x, y)$  a Euclidean shortest path in  $P$  connecting  $x$  and  $y$ .

The main goal of this paper is to show that the above guard set is sufficient to guard the whole domain. We first isolate the simple-polygon case, which will serve as a basis for the general result.

**Proposition 1** (Reflex-vertex visibility in simple polygons). *Let  $Q$  be a nonconvex simple polygon. Then every point of  $Q$  is visible from at least one reflex vertex of  $Q$ .*

**Theorem 1** (Guarding by outer reflex and hole vertices). *Let  $P$  be a polygonal domain with holes. Assume that either the outer boundary is nonconvex or  $P$  has at least one hole. Then every point of  $P$  is visible from at least one guard in*

$$G = R_{\text{out}} \cup V_{\text{holes}}.$$

Consequently,  $P$  admits a guard set of size at most  $n_r + n_h$ .

## III. PROOF OF THE MAIN RESULT

We now use Euclidean shortest paths inside the domain. The key observation is that a shortest path cannot bend in the interior of the domain or in the relative interior of a boundary edge. In a simple polygon, this implies that every interior bend occurs at a reflex vertex. In a polygonal domain with holes, it implies that every interior bend occurs either at a reflex vertex of the outer boundary or at a vertex of a hole.

We first record the simple-polygon case in a form that will also be useful on its own.

**Lemma 1** (Bends of shortest paths occur at reflex vertices). *Let  $Q$  be a simple polygon, and let  $\pi_Q(a, b)$  be a shortest path between two points  $a, b \in Q$ . Then every interior bend of  $\pi_Q(a, b)$  occurs at a reflex vertex of  $Q$ .*

*Proof.* Let  $v$  be an interior bend of  $\pi_Q(a, b)$ , and let  $u$  and  $w$  be the two vertices immediately preceding and following  $v$  on the path. Thus the path contains the two consecutive segments  $\overline{uv}$  and  $\overline{vw}$ . Since  $v$  is a bend, surely it is  $|\angle uvw| \neq \pi$  ( $\dagger$ ).

Suppose, for contradiction, that  $v$  is not a reflex vertex of  $Q$ . Then the interior angle of  $Q$  at  $v$  is at most  $\pi$ , so,  $|\angle uvw| \leq \pi$ . Hence, in a sufficiently small neighborhood of  $v$ , the polygon  $Q$  is locally convex. However, because of ( $\dagger$ ), it is also  $|\angle uvw| \neq \pi$ . Thus, consequently,  $|\angle uvw| < \pi$ . In particular, for points  $u' \in \overline{uv}$  and  $w' \in \overline{vw}$  chosen sufficiently close to  $v$ , the segment  $\overline{u'w'}$  is contained in  $Q$  (and both  $uvw$  and  $u'vw'$  are triangles).

Now replace the subpath  $\overline{u'v} \cup \overline{vw'}$  by the segment  $\overline{u'w'}$ . Since  $v$  is an interior bend of the path (and, also, both  $uvw$  and  $u'vw'$  are triangles), the points  $u', v, w'$  are not collinear. Therefore the triangle inequality is strict, and

$$|u'w'| < |u'v| + |vw'|.$$

Thus this replacement yields a shorter path from  $a$  to  $b$  contained in  $Q$ , contradicting the assumption that  $\pi_Q(a, b)$  is a shortest path. Therefore  $v$  must be a reflex vertex of  $Q$ .  $\square$

This fact immediately yields the required reflex-vertex covering property for nonconvex simple polygons.

*Proof of Proposition 1.* Let  $Q$  be a nonconvex simple polygon, and let  $p \in Q$  be arbitrary. We show that  $p$  is visible from at least one reflex vertex of  $Q$ .

If  $p$  sees all of  $Q$ , then in particular it sees every reflex vertex of  $Q$ . Since  $Q$  is nonconvex, it has at least one reflex vertex, and the claim follows.

Assume now that  $p$  does not see all of  $Q$ . Choose a point  $q \in Q$  that is not visible from  $p$ , and let  $\pi_Q(p, q)$  be a shortest path from  $p$  to  $q$  inside  $Q$ . Since  $q$  is not visible from  $p$ , the path  $\pi_Q(p, q)$  is not the segment  $\overline{pq}$ , so it has at least one bend. By Lemma 1, every interior bend of  $\pi_Q(p, q)$  occurs at a reflex vertex of  $Q$ .

Let  $r$  be the first bend of  $\pi_Q(p, q)$  after  $p$ . Then the first link of the path is the segment  $\overline{pr} \subseteq Q$ , so  $r$  is visible from  $p$ . Since  $r$  is a reflex vertex of  $Q$ , the point  $p$  is visible from at least one reflex vertex.

As  $p \in Q$  was arbitrary, every point of  $Q$  is visible from at least one reflex vertex of  $Q$ .  $\square$

We now pass to polygonal domains with holes.

**Lemma 2** (Bends of shortest paths in polygonal domains). *Let  $P$  be a polygonal domain, and let  $\pi_P(a, b)$  be a shortest path between two points  $a, b \in P$ . Then every interior bend of  $\pi_P(a, b)$  occurs either at a reflex vertex of the outer boundary or at a vertex of a hole.*

*Proof.* Let  $v$  be an interior bend of  $\pi_P(a, b)$ , and let  $u$  and  $w$  be the two vertices immediately preceding and following  $v$  on the path.

We first show that  $v$  cannot lie in the interior of  $P$ . If it did, then a sufficiently small neighborhood of  $v$  would be contained in  $P$ . Choosing points  $u' \in \overline{uv}$  and  $w' \in \overline{vw}$  sufficiently close to  $v$ , the segment  $\overline{u'w'}$  would also be contained in  $P$ . Since the path bends at  $v$ , the points  $u', v, w'$  are not collinear, and therefore

$$|u'w'| < |u'v| + |vw'|.$$

Replacing the subpath  $\overline{u'v} \cup \overline{vw'}$  by  $\overline{u'w'}$  would produce a shorter path from  $a$  to  $b$  contained in  $P$ , contradicting minimality of  $\pi_P(a, b)$ .

Next,  $v$  cannot lie in the relative interior of a boundary edge of  $P$ . Indeed, in a sufficiently small neighborhood of such a point, the domain  $P$  coincides locally with one side of a supporting line. Hence the same local shortening argument applies: nearby points  $u' \in \overline{uv}$  and  $w' \in \overline{vw}$  can be joined by

a straight segment  $\overline{u'w'} \subseteq P$ , and the strict triangle inequality again yields a contradiction.

Therefore  $v$  must be a boundary vertex of  $P$ .

If  $v$  is a vertex of a hole, then the claim follows immediately. Otherwise,  $v$  lies on the outer boundary. Suppose, for contradiction, that  $v$  is not a reflex vertex of the outer boundary. Then the interior angle of  $P$  at  $v$  is at most  $\pi$ , so the domain is locally convex at  $v$ . However,  $v$  is also a bend, so the interior angle of  $P$  at  $v$  is strictly lower than  $\pi$ . As above, choosing points  $u' \in \overline{uv}$  and  $w' \in \overline{vw}$  sufficiently close to  $v$ , the segment  $\overline{u'w'}$  is contained in  $P$ , and replacing the corner at  $v$  by this segment produces a shorter path. This contradicts the minimality of  $\pi_P(a, b)$ .

Hence every interior bend of  $\pi_P(a, b)$  occurs either at a reflex vertex of the outer boundary or at a vertex of a hole.  $\square$

We can now prove the main theorem.

*Proof of Theorem 1.* Let  $P$  be a polygonal domain satisfying the assumptions of the theorem, and let

$$G := R_{\text{out}} \cup V_{\text{holes}}.$$

We show that every point  $p \in P$  is visible from at least one guard in  $G$ .

Fix an arbitrary point  $p \in P$ . If  $p$  sees all of  $P$ , then in particular it sees every point of the set  $G$ . Since, by assumption, either the outer boundary is nonconvex or  $P$  has at least one hole, the set  $G$  is nonempty, and the claim follows.

Assume now that  $p$  does not see all of  $P$ . Choose a point  $q \in P$  that is not visible from  $p$ , and let  $\pi_P(p, q)$  be a shortest path from  $p$  to  $q$  inside  $P$ . Since  $q$  is not visible from  $p$ , the path  $\pi_P(p, q)$  is not the segment  $\overline{pq}$ , and therefore it has at least one bend.

Let  $r$  be the first bend of  $\pi_P(p, q)$  after  $p$ . By Lemma 2, the point  $r$  is either a reflex vertex of the outer boundary or a vertex of a hole. Thus

$$r \in R_{\text{out}} \cup V_{\text{holes}} = G.$$

Moreover, the first link of the shortest path is the segment  $\overline{pr} \subseteq P$ , so  $r$  is visible from  $p$ .

Hence  $p$  is visible from at least one guard in  $G$ . Since  $p \in P$  was arbitrary, the set  $G$  guards  $P$ . Therefore

$$|G| = n_r + n_h \tag{1}$$

guards are sufficient.  $\square$

We record the immediate numerical consequences.

**Corollary 1.** *Let  $g(P)$  denote the minimum number of point guards sufficient to guard a polygonal domain  $P$ . Under the assumptions of Theorem 1,*

$$g(P) \leq n_r + n_h. \tag{2}$$

*Proof.* Theorem 1 provides an explicit guard set of size  $n_r + n_h$ . Therefore the minimum possible number of guards cannot exceed this value.  $\square$

**Corollary 2.** *If  $Q$  is a nonconvex simple polygon with  $n_r$  reflex vertices, then*

$$g(Q) \leq n_r. \quad (3)$$

*Proof.* This is exactly Proposition 1, interpreted as an upper bound on the minimum number of point guards. The upper bound on guards' number for simple nonconvex polygons, where  $n_h = 0$  (and  $n_r > 0$ ), also straightforwardly comes from Corollary 1, since

$$g(Q) \leq n_r + n_h = n_r + 0 = n_r.$$

□

The proof of Theorem 1 above is triangulation-free. The guard set is defined directly from the boundary structure of the domain, and the argument uses only shortest paths and local geometric properties of their bends.

#### IV. CONSEQUENCES AND COMPARISON WITH CLASSICAL BOUNDS

Since  $g(P)$  denotes the minimum number of point guards sufficient to guard a polygonal domain  $P$ , Corollary 1 yields the upper bound

$$g(P) \leq n_r + n_h$$

for polygonal domains with holes, where  $n_r$  is the number of reflex vertices of the outer boundary and  $n_h$  is the total number of vertices of all holes.

It is natural to compare this with classical art-gallery bounds stated in terms of the total number of vertices  $n$  and the number of holes  $h$ . For point guards, a standard bound is

$$g(P) \leq \left\lfloor \frac{n+h}{3} \right\rfloor, \quad (4)$$

while for vertex guards the classical upper bound is

$$g(P) \leq \left\lfloor \frac{n+2h}{3} \right\rfloor, \quad (5)$$

as stated, for example, in [3] or [6]. Our result does not replace these bounds. Rather, it provides a different type of estimate: it is triangulation-free and depends directly on the outer-boundary reflex structure and on the total boundary complexity of the holes.

To compare the two point-guard bounds, let

$$B(P) := n_r + n_h$$

denote the bound from formula (2) proved in this paper, and let

$$C(P) := \left\lfloor \frac{n+h}{3} \right\rfloor$$

denote the classical point-guard bound from formula (4).

**Lemma 3** (A sufficient condition for improvement over the classical point-guard bound). *If*

$$n_r + n_h \leq \frac{n+h}{3} - 1, \quad (6)$$

then

$$n_r + n_h < \left\lfloor \frac{n+h}{3} \right\rfloor.$$

Consequently,

$$B(P) < C(P),$$

and the present bound gives a strictly smaller number of guards than the classical point-guard estimate.

*Proof.* Assume that

$$n_r + n_h \leq \frac{n+h}{3} - 1.$$

Then

$$n_r + n_h + 1 \leq \frac{n+h}{3}$$

and therefore

$$\lfloor n_r + n_h + 1 \rfloor \leq \left\lfloor \frac{n+h}{3} \right\rfloor.$$

Since  $n_r + n_h + 1$  is an integer, it follows that

$$\lfloor n_r + n_h + 1 \rfloor = n_r + n_h + 1.$$

Hence

$$n_r + n_h + 1 \leq \left\lfloor \frac{n+h}{3} \right\rfloor.$$

Therefore

$$n_r + n_h < n_r + n_h + 1 \leq \left\lfloor \frac{n+h}{3} \right\rfloor,$$

which proves the desired strict inequality  $B(P) < C(P)$ . □

The condition (6) in Lemma 3 can also be written in the equivalent form

$$n+h \geq 3n_r + 3n_h + 3.$$

Thus, the present bound is strictly smaller whenever the total quantity  $n+h$  exceeds  $3n_r + 3n_h$  by at least three.

It is also useful to rewrite this condition in terms of the non-reflex complexity of the outer boundary. Let  $n_c$  denote the number of non-reflex vertices of the outer boundary. Since

$$n = n_r + n_c + n_h,$$

the inequality

$$n+h \geq 3n_r + 3n_h + 3$$

becomes

$$(n_r + n_c + n_h) + h \geq 3n_r + 3n_h + 3$$

and, consequently,

$$n_c + h \geq 2n_r + 2n_h + 3.$$

Therefore the new bound is favorable when the outer boundary contains sufficiently many ( $n_c$ ) non-reflex vertices relative to the number of outer-boundary reflex vertices and the total number of hole vertices.

This gives the main optimization aspect of the result. If the outer boundary is large but only mildly nonconvex, and if the

holes are few or, in particular, have small total complexity, then the quantity  $n_r + n_h$  may remain substantially below  $\lfloor \frac{n+h}{3} \rfloor$ . On the other hand, if the outer boundary has many reflex vertices or if the holes are numerous and, particularly, complex, then the classical bound may be smaller.

The preceding comparison can be complemented by a simple asymptotic observation showing that the new bound is genuinely favorable on some infinite classes of instances.

**Proposition 2** (Existence of infinite families with strict improvement). *Assuming the previous notation, there exist infinite families of polygonal domains with holes for which*

$$n_r + n_h < \left\lfloor \frac{n+h}{3} \right\rfloor.$$

*Proof.* Take any family of polygonal domains in which  $n_r + n_h \leq k$  for some fixed constant  $k \in \mathbb{N}$ , while  $n \rightarrow \infty$ . For example, one may keep the holes and the reflex structure fixed and replace a portion of the outer boundary by an increasingly fine strictly convex chain, thereby adding arbitrarily many non-reflex vertices without creating new reflex vertices.

Then  $n_r + n_h$  remains bounded by  $k$ , so

$$n_r + n_h \leq k, \quad (7)$$

whereas  $\lfloor \frac{n+h}{3} \rfloor$  can be made arbitrarily large as  $n$  increases. In particular, as soon as

$$n + h \geq 3k + 3,$$

we have

$$\left\lfloor \frac{n+h}{3} \right\rfloor \geq \left\lfloor \frac{3k+3}{3} \right\rfloor = k+1. \quad (8)$$

Therefore

$$n_r + n_h \stackrel{(7)}{\leq} k < k+1 \stackrel{(8)}{\leq} \left\lfloor \frac{n+h}{3} \right\rfloor,$$

which proves the claim.  $\square$

The dependence on  $n_r$  in the simple-polygon case is not merely an artifact of the proof: there exist infinite families for which  $n_r$  guards are necessary.

**Proposition 3** (Tightness with respect to  $n_r$  for simple polygons). *There exist infinite families of nonconvex simple polygons  $Q$  for which at least  $n_r$  guards are necessary. Consequently,*

$$g(Q) = n_r$$

*for these families.*

*Proof.* Consider a family of simple polygons constructed from repeated vertex triples  $(a_i, b_i, r_i)$ , where  $a_i$  and  $b_i$  are convex vertices and  $r_i$  is a reflex vertex; see Fig. 2. Each triple forms a narrow tooth-like visibility pocket near the convex vertex  $b_i$ , positioned behind the corresponding reflex corner at  $r_i$ .

The construction is chosen so that, for each  $i$ , there exists a point  $x_i$  in the tooth near  $b_i$  that is visible only from points lying inside the corresponding angle  $\angle a_i b_i r_i$ , in particular

from  $r_i$ . Consequently, any guard set must contain a guard at a point lying inside this angle in order to see  $x_i$ .

Since these corresponding regions are pairwise disjoint for different indices  $i$ , at least  $n_r$  guards are necessary. On the other hand, Corollary 2 states that  $n_r$  guards always suffice for a nonconvex simple polygon. Therefore  $g(Q) = n_r$  for this family.  $\square$

The corresponding question for polygonal domains with holes seems more delicate, since the role of the hole vertices  $n_h$  may depend more sensitively on the geometry of the holes. For this reason, we restrict the present tightness discussion to the simple-polygon component  $n_r$  of the bound, and we do not pursue tightness of the  $n_h$ -term here.

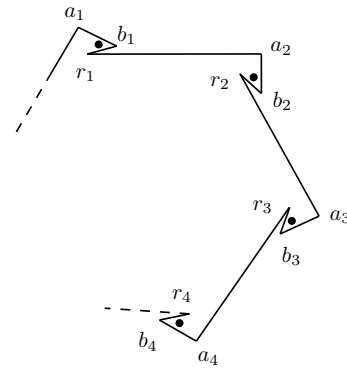


Fig. 2. A shuriken-like family of simple polygons built from repeated triples  $(a_i, b_i, r_i)$ , where  $a_i$  and  $b_i$  are convex vertices and  $r_i$  is reflex. Each tooth near  $b_i$  contains a marked witness point  $x_i$  (shown as a black dot) that is visible only from points lying in the corresponding angle  $\angle a_i b_i r_i$ , in particular from the reflex vertex  $r_i$ , and not from any other reflex vertex. Hence each such tooth requires its own guard, so at least  $n_r$  guards are necessary. Together with the upper bound  $g(Q) \leq n_r$ , this shows that the bound  $g(Q) = n_r$  is tight for this family.

A normalized visualization of Lemma 3 is obtained by considering the quantities

$$\frac{n_r}{n} \quad \text{and} \quad \frac{n_h}{n},$$

plotted on the horizontal and vertical axes, respectively, of a two-dimensional orthogonal coordinate system. Dividing the sufficient condition (6),

$$n_r + n_h \leq \frac{n+h}{3} - 1,$$

by  $n$ , we obtain

$$\frac{n_r}{n} + \frac{n_h}{n} \leq \frac{1}{3} + \frac{h}{3n} - \frac{1}{n} = \frac{1}{3} + \frac{h-3}{3n}. \quad (9)$$

Thus, in normalized coordinates, the region in which the present bound is guaranteed to be strictly smaller than the classical point-guard bound is a half-plane below a straight line. Its position depends on the ratio  $\frac{h}{n}$  and on the finite-size correction  $\frac{1}{n}$ , so the figure should be interpreted as a schematic illustration of a family of threshold lines rather than as a single universal boundary.

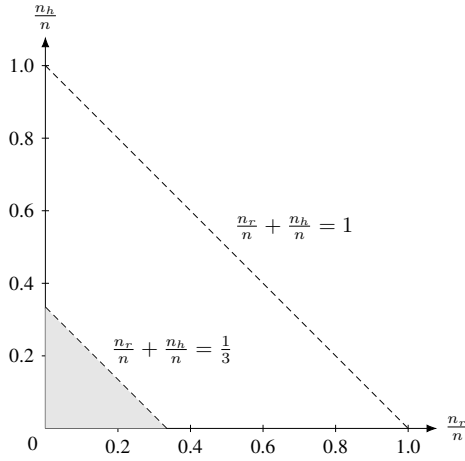


Fig. 3. Normalized comparison of the present bound and the classical point-guard bound. The dashed line  $\frac{n_r}{n} + \frac{n_h}{n} = 1$  marks the boundary of the triangular region containing the feasible parameter values, since  $n_r$  and  $n_h$  count disjoint subsets of the  $n$  boundary vertices. For  $h \geq 3$ , the shaded region below  $\frac{n_r}{n} + \frac{n_h}{n} = \frac{1}{3}$  is guaranteed to satisfy  $B(P) < C(P)$ .

In particular, if  $h \geq 3$ , then the right-hand side of (9) is at least  $\frac{1}{3}$ , so the half-plane below the line

$$\frac{n_r}{n} + \frac{n_h}{n} = \frac{1}{3}$$

is a guaranteed subset of the improvement region, that is, a region in which points with coordinates  $(\frac{n_r}{n}, \frac{n_h}{n})$  correspond to polygonal domains for which the proposed upper bound on the number of guards is strictly smaller than the classical one. By contrast, for  $h \in \{0, 1, 2\}$ , the sufficient threshold lies strictly below  $\frac{1}{3}$ , and therefore the same schematic region can no longer be guaranteed.

Fig. 3 illustrates this normalized comparison. Since  $n_r$  and  $n_h$  count disjoint subsets of the  $n$  boundary vertices,

$$\frac{n_r}{n} + \frac{n_h}{n} \leq 1.$$

Thus, the feasible parameter region is contained in the triangle below the line  $\frac{n_r}{n} + \frac{n_h}{n} = 1$ , while the shaded region shows the guaranteed improvement region described above.

## V. CONCLUSIONS

We proved that a polygonal domain with holes can be guarded by placing point guards at all reflex vertices of the outer boundary and at all vertices of the holes. This yields the upper bound  $n_r + n_h$ , where  $n_r$  is the number of reflex vertices of the outer boundary and  $n_h$  is the total number of hole vertices. The proof is triangulation-free and is based on Euclidean shortest paths and the structure of their bends.

The bound is not intended to replace classical art-gallery bounds stated in terms of the total number of vertices  $n$  and the number of holes  $h$ . Rather, it provides an alternative

structural estimate. In particular, it can be strictly smaller than the classical point-guard bound  $\lfloor \frac{n+h}{3} \rfloor$  for domains whose outer boundary has few reflex vertices and whose holes have low total boundary complexity. This was illustrated both analytically and by an infinite-family argument. In addition, we observed that the  $n_r$ -dependence is already tight for suitable infinite families of simple polygons.

Possible directions for further work include identifying sharper sufficient conditions under which the bound  $n_r + n_h$  improves on classical estimates, studying whether a similar approach can be adapted to restricted classes of domains such as orthogonal domains, and investigating algorithmic aspects of computing or reducing such structurally defined guard sets.

## VI. ACKNOWLEDGMENT AND DISCLOSURE

This paper was supported by grant IG410035, No. F4/51/2025, provided by the Internal Grant Agency of the Prague University of Economics and Business. During the preparation of this work, the author used ChatGPT 5.4, but only for language editing. All formal ideas, methodology, results, and conclusions presented in this paper are original to the author, except in the introductory part where prior work is explicitly identified and properly cited. The author takes full responsibility for the content of this publication.

## REFERENCES

- [1] V. Chvátal, "A combinatorial theorem in plane geometry," *Journal of Combinatorial Theory, Series B*, vol. 18, no. 1, pp. 39–41, 1975.
- [2] S. Fisk, "A short proof of chvátal's watchman theorem," *Journal of Combinatorial Theory, Series B*, vol. 24, no. 3, p. 374, 1978.
- [3] J. O'Rourke, *Art Gallery Theorems and Algorithms*. New York: Oxford University Press, 1987.
- [4] T. C. Shermer, "Recent results in art galleries (geometry)," *Proceedings of the IEEE*, vol. 80, no. 9, pp. 1384–1399, 1992.
- [5] F. Hoffmann, M. Kaufmann, and K. Kriegel, "The art gallery theorem for polygons with holes," in *Proceedings of the 32nd Annual Symposium on Foundations of Computer Science*, 1991, pp. 39–48.
- [6] I. Björling-Sachs and D. L. Souvaine, "An efficient algorithm for guard placement in polygons with holes," *Discrete & Computational Geometry*, vol. 13, no. 1, pp. 77–109, 1995.
- [7] A. Aggarwal, "The art gallery theorem: Its variations, applications, and algorithmic aspects," Ph.D. dissertation, The Johns Hopkins University, Baltimore, MD, 1984.
- [8] L. Štěpánek, F. Habarta, I. Malá, and L. Marek, "A lower bound for proportion of visibility polygon's surface to entire polygon's surface: Estimated by art gallery problem and proven that cannot be greatly improved," in *Proceedings of the 18th Conference on Computer Science and Intelligence Systems*, ser. Annals of Computer Science and Information Systems, vol. 35, 2023, pp. 1229–1233.
- [9] —, "A proportion of visibility polygon's surface to the entire polygon's surface: A lower bound of the proportion derived for general polygons of any shape and orthogonal polygons," in *Recent Advances in Computational Optimization*, ser. Studies in Computational Intelligence. Cham: Springer Nature Switzerland, 2025, pp. 73–98.
- [10] L. Štěpánek, "Improved upper bounds on the shortest watchman route in simple polygons: Dependence on reflex vertices and triangulation strategies," in *Proceedings of the 20th Conference on Computer Science and Intelligence Systems*, ser. Annals of Computer Science and Information Systems, vol. 43, 2025, pp. 789–794.
- [11] S. Alipour, "On guarding polygons with holes," *arXiv preprint*, vol. arXiv:2102.10317, 2021.