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Evolution of Approximation Spaces in Rough Sets: From Relational Systems to Complex Adaptive Systems

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***Abstract*—We introduce Dynamic Approximation Spaces (DAS), which greatly expand upon the approximation spaces used in the rough set approach thus far. DAS are based on the Interactive Granular Computing (IGrC) model. The fundamental principle of DAS's dynamic semantics is that they are modeled within relevant components of intelligent systems (IS), which are Complex Adaptive Systems (CAS). ISs are AI systems based on the IGrC model, particularly those that deal with complex phenomena. The fundamental entities of IGrC are complex granules (c-granules, in short). These c-granules interact in abstract and physical spaces. In processes of granulation and de-granulation, new granules emerge as computational building blocks for cognition. In other words, granules that support decision-making processes are discovered. The IGrC model differs from the Turing model in that it deals with both abstract and physical objects. Our research aims to lay the groundwork for designing and analyzing ISs in various areas. ISs can be treated as Systems of Systems (SoS) and also as special c-granulas with control. Reported research shows that DAS are objects that develop the foundation of c-granule approximation processes, which play a fundamental role in ISs. Specifically, this concerns IS processes that aim to discover high-quality approximate solutions to problems that ISs should solve.**

***Index Terms*—(interactive) granular computing (IGrC), complex granule (c-granule); control of c-granule; rough sets in IGrC; dynamic approximation space in IGrC; approximation of c-granule; complex adaptive system (CAS).**

I. Introduction

THIS article provides an overview of how the concept of the approximation space has evolved within the Rough Set (RS) approach [1]–[3], from its initial introduction by Professor Zdzisław Pawlak to dynamic models based on granular computations of complex granules (c-granules, in short) in Interactive Granular Computing (IGrC) (see, *e.g.*, [4]–[10] and https://dblp.org/pid/s/AndrzejSkowron.html).

IGrC is being developed as a foundation for Intelligent Systems (ISs) treated here as AI systems that deal with complex phenomena. ISs can be considered in IGrC as examples of c-granules with control. Their behavior depends on interactions with complex phenomena. Hence, according to opinions of experts (see, *e.g.*, [11]) classical mathematical modeling is not satisfactory for modeling the dynamics of such entities. We propose modeling such ISs as Systems of Systems (SoS) [12], [13] composed out of other, in particular Complex Adaptive Systems (CAS) (see, *e.g.*, [14]–[17]) based on the IGrC model.The dynamics of these CAS should be often considered in the context of containing or interacting with them SoS. To be successful in applications, such systems must be able to continuously interact with the physical world.

The generalization of Granular Computing (GrC) (see *e.g.*, [18], [19]) to IGrC was proposed to support the design of ISs dealing with complex phenomena. Complex phenomena involve numerous interrelated variables, elements, or factors that interact with one another in unpredictable and non-linear ways. One should consider uncertain characteristics of their elements and relationships. They are frequently challenging to fully explain or anticipate. They cannot be understood by analyzing their individual parts in isolation. Their behavior emerges from the collective interactions of these parts. It is worth citing here (https://www.santafe.edu/what-is-complex-systems-science):

> [...] *the way in which complex phenomena are hidden, beyond masking by space and time, is through nonlinearity, randomness, collective dynamics, hierarchy, and emergence - a deck of attributes that have proved ill-suited to our intuitive and augmented abilities to grasp and to comprehend.*

The proposed IGrC model allows for the modeling of the continuous interaction of ISs with proper fragments of the physical environment, as identified by the reasoning methods used to control the ISs (c-granules or their societies). The temporary models constructed are adapted by the ISs' control according to the differences in perceived changes in the models' predicted behavior compared to the actual behavior.

The IGrC model draws on existing partial results from various fields, including multi-agent systems, perception and action, machine learning, natural language processing, federated learning, cognitive networks, smart cities, cyber-physical systems, complex adaptive systems, systems of systems, and

embodied AI by attempting to bring them into a coherent framework with solid foundations (see, *e.g.*, [12], [17], [20]–[25]).

Granular networks represent the states of granular computations, which are steered by c-granules [10]. The aim of the c-granules' control is to construct approximate solutions to specified problems alongside generated computations. According to Professor Leslie Valiant's terminology, the sub-granules discovered and stored by the control of ISs in granular networks can be considered computational building blocks for cognition. These computational building blocks are used to construct adaptive data models that steer c-granule behavior in construction of high quality approximate solutions of problems to be solved by IS (a given c-granule or by society of c-granules). Methods of insightful reasoning should support the control of c-granules in discovering such c-granules over generated computations. Granular networks and their components (like granular spaces or interfaces) and granular computations over them or even sets of computations (defined by their properties) can be treated as more compound c-granules. The semantics of them is defined in the context of the control of c-granule (e.g., IS) and interactions with the physical world including complex phenomena and other c-granules.

Dynamic Approximation Spaces (DAS) based on IGrC, which are used to discover these computational building blocks as sub-granules of ISs, are becoming complex systems comprising many interacting components that are difficult to separate due to their interactions with each other and the physical environment. In turn, DAS are components of ISs treated as Systems of Systems (SoS). The justification of this claim and some challenges for RS based on IGrC are discussed. One important consequence of the considerations in this paper is that the semantics of dynamic c-granules, such as DAS in ISs, should be examined in the context of properly modeled CAS that are treated as components of ISs, which in turn are SoS.

In this paper, we discuss DAS from the rough set approach as complex, dynamic objects that are subsystems (complex sub-granules) of such ISs. Dynamics of DAS is considered in the context of such complex sub-granules of ISs as CAS. One should note that ISs consist other components treated as CAS used for controlling the behavior of ISs like responsible for the trustworthiness of ISs [26], AI alignment [27], AI risk assessment (elevateconsult.com/insights/ai-risk-assessment-steps-owners-remediation-planning-2026-guide/), or controlling interactions between different components of SoS [12]. The following principles are taken into account in the proposed modeling of DAS as CAS based on IGrC:

- Complex systems are co-evolving, multi-layer networks [28].
- Complex systems contain many interacting components (https://www.iese.fraunhofer.de/blog/complex-systems/).
- Complex systems are open (https://www.iese.fraunhofer.de/blog/complex-systems/).
- Complex systems require considering different levels of abstraction and their interactions (https://www.iese.fraunhofer.de/blog/complex-systems/).

According to [29], a dynamic approximation space can be considered as a complex phenomenon.

The control of c-granules implements the physical semantics of specified transformations in the physical world (see Fig. 1).

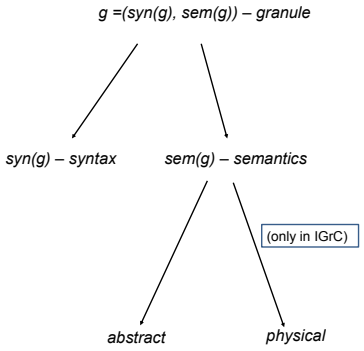


Fig. 1. Physical and abstract semantics of c-granules in IGrC.

It is important to note that c-granules in the IGrC have two meanings. The first is defined in the abstract space as part of the specification, which is the expected semantics. The second semantics is related to the physical realization of the specification through control of the c-granule and is called the real semantics. The difference between these two semantics may be attributed to unexpected interactions between the objects of physical semantics and the environment. The control of c-granule takes into account the differences between the expected and real semantics while steering granular computations.

Roughly speaking, the implementation of physical semantics of a given specification of transformation of granular network by a given c-granule is based on discovering relevant configurations of physical objects, providing the basis for perceiving relevant data about these objects and their interactions through the control of IS (c-granule or society of c-granules). Furthermore, to ensure the success of the designed ISs, the configurations of the objects adaptively change by control of IS to enable the perception of relevant data. This makes it possible to construct high-quality models on which the behavior of the ISs is based. A key property of the ISs discussed here is that they cannot be separated from their interactions with the physical world. The proposed IGrC model differs from the classical Turing model in that it synchronizes four components: language, reasoning, perception, and action [30]. In the IGrC model, granular computations form the basis for insightful reasoning methods that support problem solving by ISs (c-granules or their societies). The control of c-granule uses in its informational layer, its language, which evolves during the control behavior, particularly to express transformations that establish interactions with its second physical

layer. Note that the dynamics of the IGrC-based DAS are not confined to an abstract space. Moreover, due to the uncertainty of perceiving physical objects and their interactions with the environment, the expected results of realized transformations by the control of ISs may differ from reality. ISs should take these differences into account by adapting their behavior accordingly.

According to the terminology used by Professor Leslie Valiant (see http://people.seas.harvard.edu/~valiant/researchinterests.htm), the control of ISs (c-granules or the society of c-granules) aims to generate sub-granules that are treated as computational building blocks for cognition. In other words, these sub-granules support the control aiming to understand the perceived situation well enough to make the right decisions. The processes leading to generation of these computational building blocks, as special sub-granules, are complex and we propose to ground them on IGrC. These blocks are particularly useful for approximating complex, vague concepts that trigger transformations (*e.g.*, actions or plans) used to satisfy the needs (or solve the specified problem) of the considered IS (c-granule). In this paper, we argue that searching for these sub-granules should be carried out within properly modeled CAS of ISs. Specifically, the processes involved in discovering the basic components related to granular calculi, representing possible universes of granules, should be considered in the context of granular spaces consisting of, *e.g.*, c-granules representing adaptation strategies and strategies for discovery of new relevant relations in granular calculi as well as rules for inferring properties of the c-granules on the current hierarchical level from properties of c-granules from preceding levels. In turn, granular spaces cannot be separated from more complex structures called granular networks, which are related, *e.g.*, to hierarchical learning. Next, granular networks are considered in the context of granular computations, and ultimately, in the context of the control of ISs (c-granules or societies of c-granules). These networks are composed of many modules (sub-granules) that continuously interact in both the abstract and physical worlds. The future state of a component cannot be determined in isolation because it depends on the states of other elements, particularly those with which it interacts.

The main search mechanisms for relevant computational building blocks rely on multilevel granulation and degranulation of pre-generated c-granules. These processes are used to discover and adapt of advanced dynamic approximation spaces necessary for modeling ISs. The sub-granules must be modeled within the context of properly modeled CAS with many non-separable components (compare, *e.g.*, [31]). These systems enable the construction, discovery, or induction of sub-granules as computational building blocks that support steering granular computations to satisfy the needs or goals of ISs. This is achieved through the discovery of sub-granules that approximate concepts that trigger actions or plans that aim to construct high-quality, approximate solutions to problems that will be solved during granular computations in order to meet needs.

Problem solving (or decision support) by c-granules, particularly ISs, requires an understanding of real-world situations consisting of configurations of interacting objects. Therefore, steering granular computations with c-granules necessitates the capacity to perceive situations in the physical realm and establish connections between physical and abstract entities to document the outcomes of perception. These skills are supported by insightful reasoning over granular computations performed by c-granule control. Unlike computations based on an abstract model only, these granular computations depend on physical laws [32].

One can also relate our approach to dynamic information systems, as special c-granules with behavior dependent on the context defined by the control of c-granules. This approach differs substantially from dynamic information systems reported in the literature (see, *e.g.*, [33]–[35]), where the dynamics of information systems are specified a priori.

In summary, the novelty of the paper lies in:

- introducing dynamic approximation spaces (DAS) as a complex, dynamic object grounded in IGrC;
- modeling DAS in the context of special CAS of ISs with many non-separable components (sub-granules); among these components are granular calculi, granular spaces, granular networks, and granular computations as well as control of c-granules or societies of c-granules (ISs treated as SoS);
- realizing modeling processes of DAS through multi-level granulation and de-granulation processes;
- realizing modeling processes of DAS through the construction, discovery, or induction of sub-granules treated as computational building blocks for the cognition of perceived situations;
- using these blocks to construct approximations of complex concepts that trigger transformations (actions or plans) responsible for steering granular computations and constructing high-quality, approximate solutions to specified problems that satisfy the needs of ISs;
- providing, along granular computations, adaptation of DAS in the context of the CAS used for modeling;
- developing, the foundation of c-granule approximation processes based on DAS, particularly processes of ISs that aim to discover high-quality approximate solutions to specified problems.

II. Pawlak Model of Approximation Spaces and its Evolution Based on Relational Systems

In this section, we recall the Pawlak model [1]–[3] and using this model, we distinguish some important components of the approximation process related to it. In the abstract setting of the Pawlak model the approximation space is defined as a pair (U, R), where U is a finite set of objects and $R \subseteq U \times U$ is an equivalence (indiscernibility) relation. This relation defines a partition of U into equivalence classes called indiscernibility classes. These classes $[x]_R$ (for $x \in U$) are elementary (atomic) granules. The aggregation operation on granules produces new granules from families of elementary

granules by taking the union of them. One should note that in the Pawlak model, the universe U is treated as a given set, and the issues how objects from this set are perceived in the physical world are outside of this model. Moreover, the relation R is also treated as given a priori.

Together with the indiscernibility relation, one considers a partition of U into decision classes (decision granules). Let us denote by R_d the corresponding equivalence relation associated with this partition. In this way, we obtain a triple $AS = (U, R, R_d)$ called an approximation space. For any decision granule $X \subseteq U$, its lower approximation relative to R is defined by

$$LOW(AS, X) = \{x \in U : [x]_R \subseteq X\},$$

and its upper approximation is defined by

$$UPP(AS, X) = \{x \in U : [x]_R \cap X \neq \emptyset\}.$$

The set

$$Bd(AS, X) = UPP(AS, X) \setminus LOW(AS, X)$$

is called the boundary region of X relative to R. If $Bd(AS, X) \neq \emptyset$, then X is called rough (relative to R); otherwise, it is crisp (relative to R).

Hence, one can observe that in this model we deal with two simple granular spaces, G_R and G_{R_d}, corresponding to the partitions defined by relations R and R_d : $G_R = (\{[x]_R\}_{x\in U}, agg)$, where $agg : P(\{[x]_R\}_{x\in U}) \to P(U)$ and $agg(Z) = \bigcup Z$ for $Z \subseteq \{[x]_R\}_{x\in U}$, and $G_{R_d} = U/R_d$, where $U/R_d = \{[x]_{R_d}\}_{x\in U}$. The first component of G_R contains a family of elementary granules defined by R, and the second component is an aggregation operation generating definable granules from elementary granules. In the case of G_{R_d}, we have only one component, namely the family of decision granules.

The granular spaces[1] G_R, G_{R_d} (see Fig. 2) are linked by an interface $Inter(G_R, G_{R_d})$ including certain relations between granules from these two granular spaces, as well as some reasoning methods that make it possible to find relevant relationships between granules from these spaces.

In the considered example, this interface has a simple form and is defined by

- the set-theoretical inclusion relation between definable granules from G_R and granules from G_{R_d},
- the set-theoretic 'non-empty intersection' relation between definable granules from G_R and granules from G_{R_d} and
- Res – reasoning module (RM) consisting of reasoning tools supporting, *e.g.*, the definition of approximation regions such as lower approximations of decision classes or boundary regions; the construction of search heuristics for maximal (with respect to set inclusion) definable granules included in decision granules for the lower approximation of decision granules and minimal (with respect to set inclusion) definable granules including decision granules for upper approximations.

[1]In the example, granular spaces are reduced to simple granular algebras.

Two simple granular spaces connected by an interface form a basic example of a granular network. In general, however, the structures of granular networks may be much more complex. Specifically, sets of granules in granular spaces can be defined by granular algebras, which makes it possible to generate granules from atomic ones using granulation and degranulation operations like union, intersection, or generalization. The universe of granules can be bounded by constraints that specify the context in which the granules should be considered. Furthermore, new granules can be defined in a language whose interpretation allows one to define the semantics of granules. Moreover, semantics of some granules are defined by physical semantics realized by the control of c-granule [10]. It should be noted that the granular algebra, language, and interpretation should be derived from data and knowledge bases through interaction with experts and/or chatbots, *e.g.*, through dialogue or communication. Together, they create what is called a granular calculus (see Fig. 3). Therefore, granular spaces should be considered in the context of discovery and/or adaptation strategies for such entities, granular networks (which can have nested structure Fig. 4) and the control of ISs (c-granules or their societies). These components interact with abstract and physical spaces within a complex adaptive system that models them dynamically. Let us note that the components of granular calculi must be discovered by ISs from data or knowledge bases or in dialogues with experts and/or chatbots. In particular, this concerns the discovery and adaptation of language. Hence, granular calculi should be considered in the context of other components (sub-granules) providing, *e.g.*, discovery of adaptation strategies.

In the considered example (see Fig. 5, two granular spaces G_R and G_d are shown. For simplicity, only elementary granules defined by indiscernibility classes are considered. For granules $g_x = (syn(g_x), sem(g_x))$, where $x \in U$, $syn(g_x)$ is the name of the indiscernibility class defined by x (using any bijection $f : U/R \to \{1, \ldots, \mid U/R \mid\}$), and $sem(g_x)$ is the indiscernibility class defined by x. In the interface, two relations are considered, namely the inclusion relation between the semantics of granules and the hitting relation, which makes it possible to check whether the semantics of two granules have a non-empty intersection. Res consists of mechanisms for reasoning leading to the construction of approximation regions corresponding to the lower approximation, upper approximation, and boundary region of decision classes.

In Fig. 5, labeled links connect atomic granules and indicate two types of relationships: inclusion or partial inclusion. In general, these labeled links may represent facts about much more complex relationships between granules from different, more advanced granular spaces. For instance, information granules may contain details about the location and time of a measurement stored in their informational layers. They may also concern properties of various segments of multi-time-series data or the aggregation of such data into clusters or more complex structural objects. Notably, these granules may also

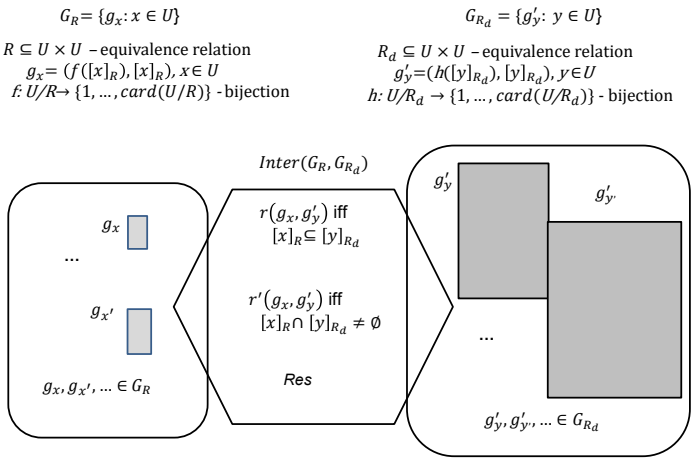
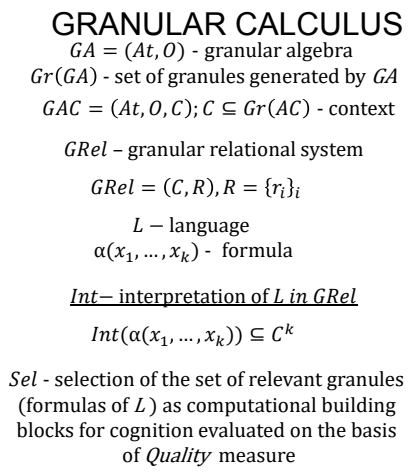


Fig. 2. Illustrative example of simple granular network for the Pawlak rough set model.



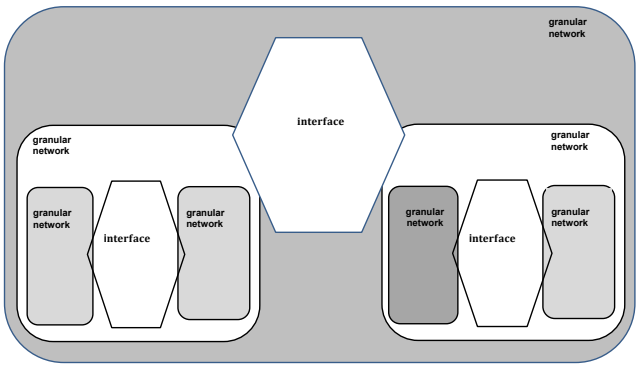


Fig. 4. Network with nested structure.

Fig. 3. Granular calculus structure. It is discovered and adapted from data and/or knowledge bases in cooperation with users, experts, chatbots. Relations discovered use both the abstract and physical semantics. Note that also quality measures are represented in the granular relational system.

include specifications for so-called associations (specifications of transformations), which enable the control of IS's (c-granules equipped with control) to generate their meaning in interaction with the physical world. Links may even represent time-dependent relationships between different granules or their parts.

Let us now turn to explanation of granulation and de-granulation processes involved in the process of construction of approximation regions. The network in Fig. 5 is built from two granular spaces: G_R and G_{R_d}. These spaces can be treated as representations of the R_d, i.e., specification of the decision information granule to be approximated, as well as a granule that provides data or knowledge conditional attributes in the form of the R relation and a method for evaluating the quality of the approximation. The first step in the approximation process involves de-granulating R_d and R into equivalence classes, which are defined by the decision d and conditional attributes, respectively. Next, interaction occurs between these equivalence classes, resulting in information granules that provide information about the relationships between the equivalence classes (Fig. 5). Finally, these information granules about the relationships between the classes are used to de-granulate the equivalence classes of R into the approximation regions of R_d, and the quality of the approximation is estimated using a selected quality measure. This is a simple illustration of the approximation processes performed by the control of ISs (c-granules or societies of c-granules). This process is much more complex in the case of more advanced applications and should be performed in the context of a complex adaptive system responsible for generating such processes. In particular, note that the entire process in the presented example occurs in an abstract space, whereas in more advanced applications, interactions with the physical space become unavoidable.

Specifications may take a form similar to that of information granules, but they may also take the form of association

$G_R = \{g_x : x \in U\}$

$R \subseteq U \times U$
$g_x = (f([x]_R), [x]_R), x \in U$
$f: U/R \rightarrow \{1, \ldots, |U/R|\}$ - bijection

$G_{R_d} = \{g'_y : y \in U\}$

$R_d \subseteq U \times U$
$g'_y = (h([y]_{R_d}), [y]_{R_d}), y \in U$
$h: U/R_d \rightarrow \{1, \ldots, \lceil U/R_d \rceil\}$ - bijection

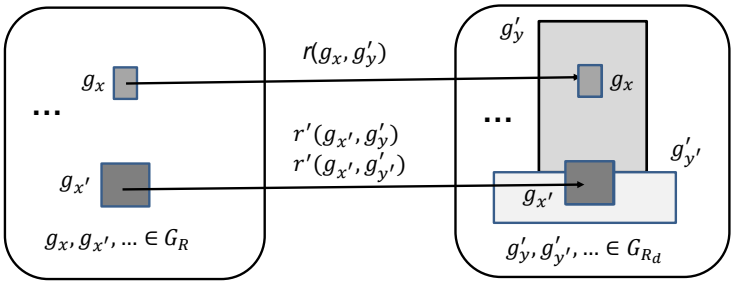


Fig. 5. Facts expressing relationships between granules from granular spaces in Figure 5 stored in the interface.

specifications. In the case of an association, the predicted results of physical realization may differ from the actual ones obtained through perception in the physical world. Therefore, the control of a c-granule should be aware of this fact and should adapt the currently used complex game or modify the target goals accordingly if such modification is necessary because the generated computation would otherwise fail to achieve satisfactory quality.

In this way, granular computations have states represented by granular networks and the discovered links between their components. The IS control (control of c-granules) specifies the transitions realized between states. In the final state of granular computation for the discussed example, all atomic granules are linked to decision granules with corresponding information about the relationships, and approximations of decision granules are computed. For example, atomic granules from G_R linked to at least two decision granules from G_{R_d} belong to the boundary region.This simple example illustrates an important aspect of approximation processes running over structures defined by granular networks. In the discussed example, approximations of decision granules play the role of target granules. In different applications, other complex high-quality objects may need to be constructed, *e.g.*, learning algorithms, classifiers, clusters, complex physical or abstract objects, or even entire granular computations possessing required properties.

Adaptation of approximation spaces is performed along granular computations (see Fig. 6).

Let us now consider two simple example illustrating how atomic granules may be obtained.

The first example illustrates how atomic granules may be obtained from sensory measurement. A specification of measurement specification is realized by the control of IS (c-granule or society of c-granules) in the physical space [10] and encodes in its informational layer the results of measurements using an information system $IS = (U, A)$ with attributes $a : U \rightarrow V_a$, where V_a is the value set of attribute a. We assume that for each attribute a, a relational structure $Rel_a = (V_a, R_a)$ is given, where R_a is a set of relations over V_a, *i.e.*, for $r \in R_a$, $r \subseteq V_a^{ar(r)}$, where $ar(r)$ is called the arity of relation r. For example, V_a may be a set of real numbers and R_a may contain the linear order relation over V_a. Moreover, together with the relational structure Rel_a, we consider a set of formulas F_a and its semantics $sem : F_a \rightarrow P(U)$ assigning to each $\alpha \in F_a$ its meaning $sem(\alpha) \subseteq U$. New atomic granules representing the results of measurements are of the form $(\alpha, sem(\alpha))$ for all attributes $a \in A$, the corresponding families F_a, and semantics sem.

Our previous example was pertained to sensory granules. The example we are going to present is related to discovering new atomic granules from already defined ones on the current hierarchical level. This example illustrates the necessity of considering features of situations in the context of, *e.g.*, a defined time window around the current moment. Let us consider a granular space related to an information system $IS = (U, A)$ with the set of granules G defined by indiscernibility classes of $IND(A)$ and an empty set of relations between granules. We assume that there is one distinguished attribute $t \in A$ representing time, with value set $V_t = \{1, \ldots, n\}$. We define atomic granules for a new granular space GS', using an aggregation operation. Any new atomic granule is a time window of length T, *i.e.*, a sequence of length T of terms being value vectors of attributes of the length T, obtained from signatures of objects from U, such that each successive term

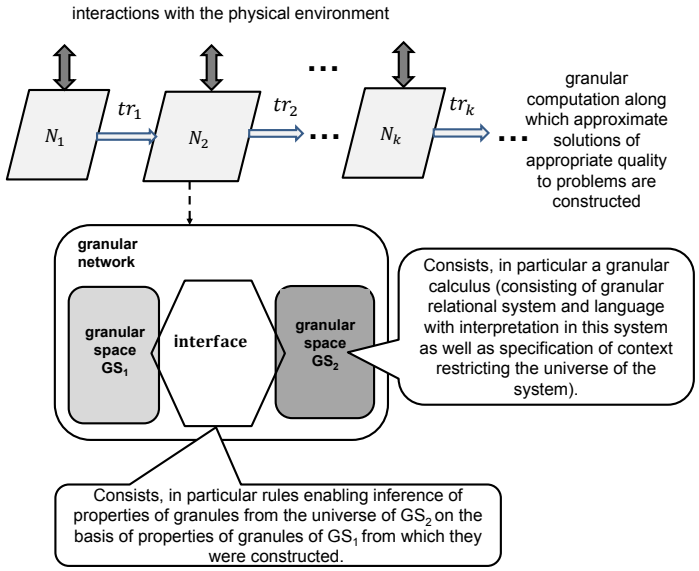


Fig. 6. Adaptation of approximation spaces along granular computations.

in the sequence has a value of t greater by one than that of the preceding term. Certainly, one may generalize this example by considering more than one operation or by considering new granules obtained by applying operations a finite number of times to IS or to already generated granules.

Granular networks are modified by the control mechanism of granules in order to achieve specific target goals. On the basis of the current granular network, the control selects a transformation according to its rule set (complex game) in order to change the current network into a new granular network representing the next step of granular computation. This control aims to generate granular computations over granular networks that satisfy a given specification to a satisfactory degree. This specification may concern either the final state of the computation or the entire computation, depending on the task performed by the granule.

The quality of computations is assessed using selected proper quality measures. Various such measures have been developed in Machine Learning (ML) and in different application areas, such as risk management, algorithmic trading, drug discovery, and computers based on biological computing [36]–[40]. For instance, in ML, the Minimum Description Length (MDL) principle [41] and various measures based on the confusion matrix [36] are well known for estimating the quality of learning algorithms and classifiers.

In some applications, it may be necessary to preserve certain properties (*e.g.*, system "safety") represented by a granule as invariants throughout the computation over granular networks. In other cases, it may be necessary to enrich a granular network with properties expressing, *e.g.*, that a desired drug has just been synthesized under constraints related to the cost of carrying out the computation.

The reader is referred for more information on IGrC to the aforementioned papers on IGrC. Here, we summarize the already presented comments about IGrC and add new comments on IGrC and ISs.

In IGrC, we consider complex granules (c-granules) that connect objects from both abstract and physical realms. It is worth mentioning that IS's can be viewed as compound c-granules equipped with control. The extension of GrC to IGrC is essential for addressing the perception of situations in the physical world by IS's. Understanding perceived situations in the physical world is crucial for IS's, especially those dealing with complex phenomena, in order to make appropriate decisions regarding these situations. These decisions concern selection of transformations and their realization. The implementation of transformation specifications in the physical world is used to control granular computations of the considered IS (or control of a c-granule) in such a way as to achieve the target goals of the IS.

For IS's dealing with complex phenomena, two kinds of granules should be considered: (i) information granules, defined in GrC by formulas over an appropriate language with semantics in the space of granules at a given hierarchical level, and (ii) complex granules (c-granules) in IGrC, with physical semantics realized in the real physical world, ensuring the perception of relevant properties of objects and configurations of physical objects, as well as of their interactions.

It is essential to recognize that the control of an IS must continuously interact with the physical environment in order to respond to changes in perceived situations caused by environmental interactions. Collecting data from such interactions only once is insufficient for developing a high-quality model. The created models must be adapted whenever substantial changes in physical space are perceived. In addition, the control of an IS should be equipped with appropriate reasoning techniques supporting its behavior. These issues will be

explored in greater detail in the following sections.

A. Optimization in the rough set approach

The first RS model, proposed by Pawlak (see, *e.g.*, [1]–[3], [42]), was based on two partitions of a given, finite set of objects. Conditional attributes define the first partition, and a decision attribute defines the second. The first approach was actually related to binary decision partitions defined by sets and their complements. The lower and upper approximations, as well as the boundary region of the binary decision partition, were approximately expressed in terms of the partition defined by conditional attributes. In this approach, the partitions are given a priori. Each conditional attribute defines a partition of the set of objects. The partition defined by a set of conditional attributes is the intersection of the partitions defined by each conditional attribute. In this approach, the data reduction optimization problem is as follows: Given a set of conditional attributes, find minimal sets of attributes, called reducts, that define the same partition as the original set. Note that a given set of attributes may have an exponential number of reducts and that the problem of finding a minimal reduct is NP-hard. This approach was extended od data reduction relative to decision.

The definition of an attribute value set should be considered in the context of relational systems. In the discussed case, the equality relation on the value set is considered together with the value set. Next, richer relational systems are considered. For example, when the values are reals, one can consider the natural relation on reals together with the value set. The relations in the Cartesian product of a given attribute's value set naturally define the relations in the Cartesian product of the object set. This is how the evolution of the c-granule language begins. This language is useful for expressing granules. The outlined approach was extended to numerous applications, such as the discretization of real-value attributes, symbolic value grouping, association rules, the generation of decision rules, and decision support related to preferences. This can be characterized briefly as follows. Rather than considering a single partition of conditional attributes, an algebra of partitions is considered. Such an algebra consists of atomic partitions and operations on those partitions. Operations enable the creation of new partitions from both atomic partitions and previously generated ones. The sets of atomic partitions may be much richer than in the initial model. For example, one can consider relations related to the degrees of inclusion of classes of partitions [43] or relations related to quality measures of defined approximations. The latter are introduced to balance the description length with the quality of the approximation. The sets of partitions in the algebra of partitions can be large or even infinite (*e.g.*, in the case of searching for features defined by hyperplanes discerning training cases [20], [44]). This leads to more complex optimization problems. Over the years, Boolean reasoning was used to construct efficient heuristics for solving these problems (see, *e.g.*, [42], [44], [45]). The discussed approach was also extended by considering coverings instead of partitions (see, *e.g.*, [46]–[48]).

Readers familiar with Machine Learning (ML) recognize that the discussed examples are closely related to major ML problems such as feature selection or feature engineering [20], [44], [45], [49]–[51], in particular to hierarchical learning (see, *e.g.*, [52], [53]).

However, this approach proves inadequate for more advanced applications, particularly because it lacks the insightful reasoning methods necessary to effectively understand real-world situations and make the right decisions concerning, *e.g.*, searching for new data. Dynamic algebras and languages over such algebras that define new, relevant granules as computational building blocks for cognition need to be discovered. Furthermore, the entire process cannot be confined to an abstract space because continuous interaction with the physical world is necessary to understand situations and make appropriate decisions. This led to the emergence of the IGrC paradigm.

The main challenge for the control of IS is to provide reasoning mechanisms for developing efficient heuristics that search for accurate approximations of complex, vague concepts at the highest level of the hierarchy that are relevant for triggering for actions or plans. In multiscale modeling, the objective is to identify dependencies between various levels, such as the cellular and organ levels in biological organisms (see, *e.g.*, [54]).

III. RS GROUNDED IN IGRC

We generalize the RS approach was generalized by introducing approximation spaces defined over dynamic networks of granular spaces in IGrC (see, *e.g.*, [10]). The goal is to develop RS-based methods that can direct granular computations of the IS control and support the discovery of approximate solutions of an appropriate quality for specific problems. These methods facilitate the discovery of complex games within granular networks. These games consists of finite sets of models of vague, complex concepts labeled by transformations (associations) triggered when the concepts are sufficiently satisfied. Such games are used by the control of c-granules for guiding granular computations when interacting with abstract and physical worlds. These methods should enable the construction of "abstract and/or physical" approximate solutions of the appropriate quality for a given problem specification for the c-granule(s) involved in these computations. New sub-granules that interact with the physical world are discovered during computations generated by c-granules over granular networks. These sub-granules serve as essential computational building blocks for understanding perceived situations in the physical world. Along with the generated computations, c-granule control constructs approximate satisfactory solutions to the presented problems. Hence, control behavior is based on complex games. These complex games are discovered and adapted during granular computations.

From the above discussion, we can conclude that the universe of objects for which concept approximations are

considered consists of granular computations. Consider the problem of discovering granular computations that certainly satisfy the following exemplary concept:

Granular computation has a satisfactory quality relative to the task specified for a given c-granule.

Let us assume that the relevant quality measure and threshold for expressing "satisfactory quality" was specified. In RS, the aim is to discover examples of computations that belong to the lower approximation of this concept. The control of c-granule should be able to generate such examples, although examples of these computations are not yet known. Unlike in typical machine learning tasks, the set of training objects is not predetermined. Instead, the c-granule generates training cases, computations in the considered case, to meet its needs. The c-granule's control system aims to satisfy its needs (goals) by generating high-quality granular computations relative to the specifications of those needs. Quality measures express the quality of the approximate solutions constructed during computations. This problem (Fig. 7) is challenging and involves developing insightful reasoning methods to discover computational building blocks that support the construction of approximate solutions over granular computations. Despite the fact that Aristotle addressed such logic of discovery, demonstrating how the premises of a syllogism must be formulated to yield a given conclusion, one should note that the logic of discovery is still in its infancy.

Let us look on the above process of generation of relevant granular computations from the point of view of approximation of granules. In general, approximation of granules is steered by interactions of granules generated through multi-level granulation and de-granulation processes.

The main question we consider is the following: How to discover (induce) from c-granule g_1^* representing the initial information about the problem to be solved (*e.g.*, in the form of data or knowledge, constraints) the (semi-) optimal approximation g^* of a given problem specification in the form of c-granule g_2^* ?

The search for g^* is based on a multi-level process of granulation and de-granulation, starting from g_1^* and g_2^*. This process is steered by the results of the interactions between the generated granules with the use of proper quality measures[2].

Consider the approximation of a decision function represented by a c-granule, g_2^*, with a c-granule, g_1^*, representing an information system with conditional attributes. The approximation is realized by granular computation. In the simplest case, this can be organized by de-granulating g_1^* and g_2^* into sets of sub-granules (e.g., indiscernibility and decision classes, respectively) and generating interactions among these classes. This results in the generation of facts about the degree to which the interacting classes are (partially) included. Next, de-granulation of the granules from g_1^* is performed based on these facts into regions that approximate the decision function. A given measure, $Meas$, expresses the quality of the approximation (see Fig. 9).

Optimization may be necessary in selecting relevant interactions that lead to the de-granulation and granulation of the generated sub-granules. In more complex cases (*e.g.*, related to searching for the high quality approximate solutions of problems specified by g_2^* on the basis of g_1^* consisting of data and knowledge basis, providing methods of dialogues with experts or chatbots), searching for relevant approximations of specification granules may require generating multi-level complex structures consisting of, *e.g.*, granular calculi (see, Fig. 3) and granular spaces, as well as networks over them, which may need to be adapted. These structures are used to identify sub-granules as computational building blocks for cognition. They support decision-making processes that approximate the specification g_2^*.

All the granules generated in the aforementioned granulation and degranulation processes are treated as computational building blocks for cognition. These blocks support construction processes by controlling the quality of the approximate solutions to the specified problem (solutions that satisfy the needs to a satisfactory degree). In this setting, approximation spaces are becoming adaptive complex systems that aim to discover relevant computational building blocks interacting with complex phenomena in the physical world in the context of the c-granule control. Similarly to how natural language is treated as an adaptive complex system [22], models of approximation spaces should have the following properties:

- They consist of multiple c-granules (societies of c-granules) that interact with one another.
- They are adaptive, meaning c-granule behavior is based on past interactions. Current and past interactions together influence future c-granule behavior.
- The behavior of c-granules is the consequence of competing factors ranging from perceptual constraints to social motivations.

The components of approximation spaces are difficult to separate due to their interactions with each other and physical environment. The future state of a component cannot be determined in isolation because its state depends on the states of the other components with which it interacts [55]. As it was already mentioned, according to opinions of experts (see, *e.g.*, [11]) classical mathematical modeling is not satisfactory for modeling the dynamics of such entities. We propose to model this dynamics in the context of a complex adaptive system (CAS) being a sub-granule of the considered IS (c-granule or society of c-granules). This CAS has different components such as granular calculi, granular spaces, granular networks or granular computations as well as other modules of the control of IS. Granular calculi, in particular, are components of approximation spaces and should be considered in the context of granular spaces. These spaces consist of methods that are directly responsible for constructing and adapting granular calculi. In turn, granular spaces interact through interfaces that enable the transmission of information between them,

[2]Depending on the application, these measures may depend on the entire granular computation or its final state.

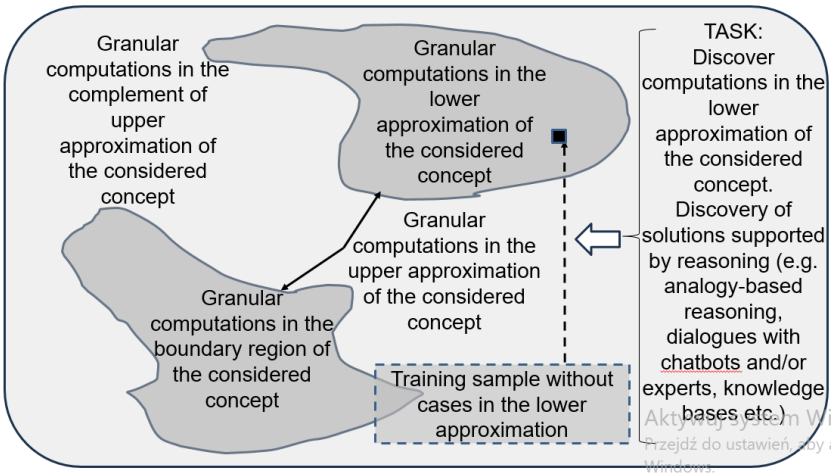


Fig. 7. Approximation areas of the concept: *Granular computations with high-quality solutions of the specified problem in the domain consisting of computations of c-granule control.*.

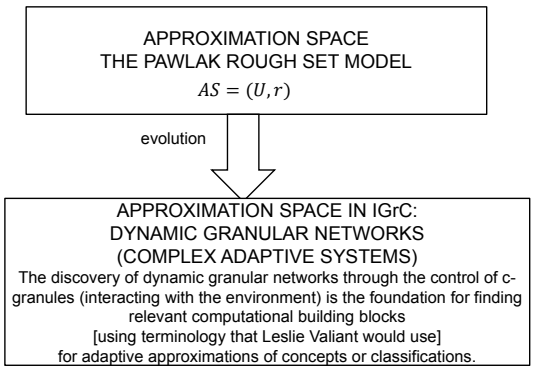


Fig. 8. Evolution of approximation spaces.

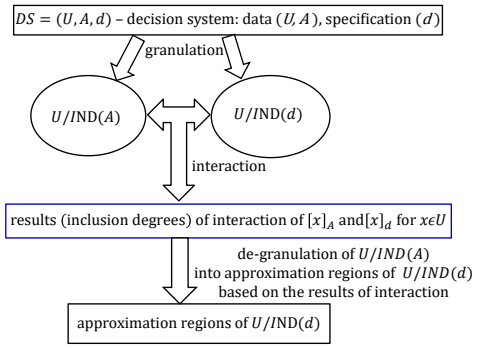


Fig. 9. Approximation of granules steered by interactions of granules generated through multilevel granulation and de-granulation processes: Illustrative example.

creating granular networks. Furthermore, granular networks are considered in the context of controlling the c-granule, which steers granular computations and adapts existing ones to perceived changes in the physical environment in order to achieve the c-granule's needs. The c-granule control is composed of many sub-granules that interact with each other and with a physical environment. Among these sub-granules are modules that support the c-granule's reasoning. This reasoning should be used to find the right answers to questions such as 'Where?', 'When?', 'What?', 'Who?' and 'Why?' when searching the physical world for relevant data on which to base decisions. In the case of C-granule societies, the interactions that guide C-granule behavior involve communication between C-granules, human experts, and chatbots. These entities aim to organize their cooperation or competition.

The formalization concerning this understanding of approximation and its role in discovery of approximate problem solving by ISs (c-granules or societies of c-granules) will be elaborated in more detail in our next paper.

Here, we would like to mention that this formalizations would clarify in particular, the following issues:

- problem specification is encoded into the initial granular network of computation,
- solutions of the considered problem are expressed by granules (*e.g.*, when the task is to learn classifiers or compound physical or chemical objects),
- different fragments of granular networks are expressed using different languages and linked by interfaces (*e.g.*, based on aggregation operations between granules),
- successive steps of computation are obtained by control of c-granule interacting with the physical world (*e.g.*, by generating more compound granular networks from existing ones or extending the existing ones by granules resulting from the relevant interactions with the physical world or domain knowledge bases),
- selection of rule for transformation of the current granular network is performed,
- the rule set for transformation of granular networks is

adaptively learned,
- control can search for new data sets in the physical world for updating granular networks (by reasoning, *e.g.*, when, where to search for such data),
- different components of approximation spaces are distinguished to make it possible to define the quality of approximation of granules,
- selection of optimization procedures for approximation is performed among many possible ones,
- control of c-granule is designed to make it possible to generate computation of the required quality (*e.g.*, by optimization and adaptation of its behavior),
- quality of computation is evaluated by the relevant quality measures (*e.g.*, some of them may use the whole computation for evaluation when the task is to preserve some invariants during computation and some other ones depend only on the final granular network in the computation),
- quality of generated approximate solution is tested in the physical world.

It is worth noting that the transformation reealized by granular computations for approximating c-granules is different from that of classical algorithms. According to Valiant's opinion [56], unlike most algorithms, the new algorithms, ecorithms, can be run in environments unknown to the designer. They learn how to act effectively in the environment by interacting with it. After sufficient interaction, the algorithm will have expertise not provided by the designer but extracted from the environment. Moreover, granular computations in IGrC run in abstract and physical spaces, progressing through the interaction of abstract and physical objects in open world. The control of IS (c-granule or society of c-granules) is supported by reasoning performed in the context of CAS that create components of the IS system when steering them, in particular in adaptation of its structure. Thus, the realization of transformations in IGrC differs substantially from the realization of classical algorithms.

Proper behavior of ISs requires the development of reasoning methods that support the steering of granular computations by ISs, particularly within the context of different CASs that comprise ISs. This requires discovering reasoning and machine learning methods concerning granular computations that run in physical and abstract spaces. These methods aim to synchronize language, reasoning, perception, and action [30]. They transcend the traditional computing model, which is confined to an abstract space. Algorithms that are unknown a priori should be learned through interactions in an open world. This includes interactions with physical objects, as well as the perception of their properties and interactions. The methods of learning such algorithms transcend the statistical machine learning methods that have been used thus far. This may involve reasoning to support an understanding of perceived situations in order to make the right decisions, as well as selecting sensors and actuators as well as discovering new ones together with their application locations. Thus, this perspective is broader than Valiant's view that two conflicting characteristics must be reconciled: the logical nature of reasoning and the statistical nature of learning (https://people.seas.harvard.edu/~valiant/researchinterests.htm). These two characteristics should be substantially generalized and grounded in the proper computational model.

In the generalized approach to the rough set-based approximation of granules, reasoning performed by the control of c-granules over the generated granular computations plays a crucial role. It is worth to emphasize that the control have many different sub-granules supporting reasoning of c-granule in steering granular computations (see, *e.g.* [10] and Fig. 10)

As we discussed above, reasoning supporting approximation of granules is performed along granular computations by control of c-granule.

Fig. 11 presents a challenge for rough sets in IGrC related to discovery of approximate solutions of problems. Note that in this case, there are no provided examples belonging to the lower approximation of the concept *granular computation with satisfactory quality relative to the specified task*[3]. However, such examples should be discovered. There are numerous domains for which this challenge is important. Among them (i) discovery of learning algorithms and construction of classifiers [20], [49], (ii) automatic design of robots [57], (iii) drug discovery [39], (iv) algorithmic trading [58], (v) generative AI [59] are a few. One could request the development of common foundations based on IGrC and RS for designing and analyzing discovery systems from these different domains [60].

IV. Conclusions

We presented a new approach to approximating granules that supports the discovery of approximate problem solving. This approach is based on the IGrC model. An approximation space is defined as a dynamic object modeled in the context of Complex Adaptive Systems (CAS), which are components of IS. These CAS consist of parts such as granular calculi and granular spaces, which, when linked by multi-layered interfaces, create granular networks. Granular networks, in turn, are steered by the control of the IS (c-granule or society of c-granules) to generate granular computations. The high quality of the approximate solutions to the specified problems are constructed (discovered or induced) along these computations. Unlike current approaches, the approximation spaces and processes realized on the basis of them cannot be confined to an abstract space. These processes run through interactions with both abstract and physical spaces.

This approach is a step toward establishing the foundation for designing and analyzing AI systems. Moreover, it opens new perspectives for the research on rough sets as well as fuzzy sets [61]–[63] not restricted to the abstract space only, they "live" in the interacting abstract and physical spaces.

We will continue our research on the foundations of IGrC, aiming to extend our current findings to include societies of c-granules. This will involve further studies both on the control

[3]The lower approximation may be defined, *e.g.*, by specifying 'satisfactory quality' by the relevant threshold.

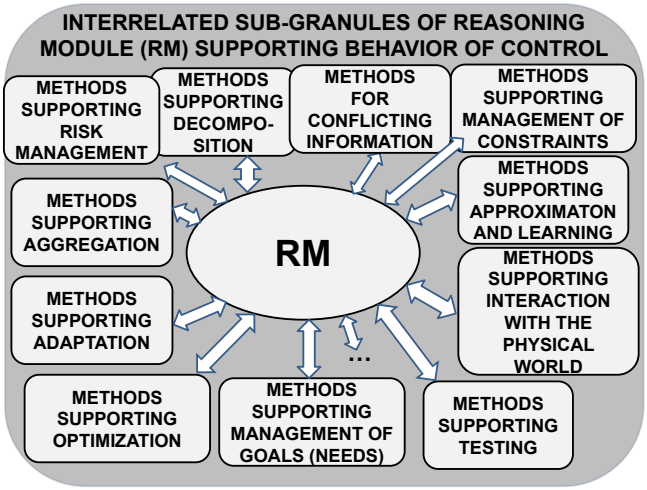


Fig. 10. Reasoning module of the c-granule controls.

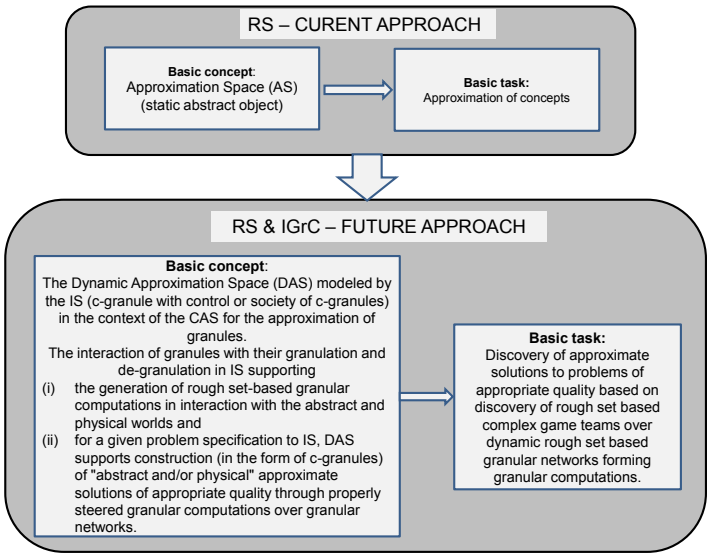


Fig. 11. A challenge for RS: discovery of approximate solutions of problems.

of c-granules and on dialogues with experts, chatbots, large language models (LLMs), and agentic systems.

One of the challenges is to develop a granulated deep learning approach on the basis of the IGrC model [64]. Moreover, combining the rough set approach grounded in IGrC with Lifelong Learning (LL) [65] is also a major challenge. LL calls for learning techniques suitable for dynamic and open worlds or environments in a self-supervised manner, which is consistent with the aims of IGrC. Many recent applications (*e.g.*, chatbots, self-driving cars, or AI systems interacting with humans and physical environments) must cope with dynamic and open environments. Hence, they should continuously learn new things in open world in order to function properly. Techniques supporting the realization of this goal should be grounded in an appropriate computing model.

The proposed IGrC model may also be treated as a step toward combining the physical structure and thinking behavior of the brain [66]:

> *We should combine the physical structure and thinking behavior of the brain, add physical priors, break through the bottleneck of computing power, realize low-power, low-parameter, high-speed, high-precision, non-depth AI models, and develop more efficient artificial intelligence technology.*

The role of solid foundations for IGrC is also crucial for the development of methods for exploring Web Intelligence 3.0 in the Social-Cyber-Physical-Thinking Space [67].

REFERENCES

[1] Z. Pawlak, “Rough sets,” *International Journal of Computer and Information Sciences*, vol. 11, pp. 341–356, 1982. [Online]. Available:

doi.org/10.1007/BF01001956

[2] ——, *Rough Sets: Theoretical Aspects of Reasoning about Data*, ser. System Theory, Knowledge Engineering and Problem Solving. Dordrecht, The Netherlands: Kluwer Academic Publishers, 1991, vol. 9. [Online]. Available: doi.org/10.1007/978-94-011-3534-4

[3] Z. Pawlak and A. Skowron, "Rudiments of rough sets," *Information Sciences*, vol. 177, no. 1, pp. 3–27, 2007. [Online]. Available: doi.org/10.1016/j.ins.2006.06.003

[4] A. Skowron and A. Jankowski, "Rough Sets and Interactive Granular Computing," *Fundamenta Informaticae*, vol. 147, no. 2-3, pp. 371–385, 2016. [Online]. Available: doi.org/10.3233/FI-2016-1413

[5] ——, "Toward W2T foundations: Interactive granular computing and adaptive judgement," in *Wisdom Web of Things (W2T)*, N. Zhong, J. Ma, J. Liu, R. Huang, and X. Tao, Eds. Heidelberg: Springer, 2016, pp. 47–71. [Online]. Available: doi.org/10.1007/978-3-319-44198-6_3

[6] A. Skowron, A. Jankowski, and S. Dutta, "Interactive granular computing," *Granular Computing*, vol. 1, pp. 95–113, 2016. [Online]. Available: doi.org/10.1007/s41066-015-0002-1

[7] A. Jankowski, *Interactive Granular Computations in Networks and Systems Engineering: A Practical Perspective*. Heidelberg: Springer, 2017. [Online]. Available: doi.org/10.1007/978-3-319-57627-5

[8] A. Skowron and D. Ślęzak, "Rough sets turn 40: From information systems to intelligent systems," in *Proceedings of the 17th Conference on Computer Science and Intelligence Systems, FedCSIS 2022, Sofia, Bulgaria, September 4-7, 2022*, ser. Annals of Computer Science and Information Systems, M. Ganzha, L. A. Maciaszek, M. Paprzycki, and D. Ślęzak, Eds., vol. 30, 2022, pp. 23–34. [Online]. Available: doi.org/10.15439/2022F310

[9] ——, "Rough sets in interactive granular computing: Toward foundations for intelligent systems interacting with human experts and complex phenomena," in *Proceedings of the International Joint Conference on Rough Sets (IJCRS 2023), Cracow, October 5-8, 2023*, ser. Lecture Notes in Computer Science, vol. 14481. Berlin, Heidelberg: Springer, 2023, pp. xv–xxxvii. [Online]. Available: doi.org/10.1007/978-3-031-50959-9

[10] A. Skowron, A. Jankowski, and S. Dutta, "Interactive granular computing: Toward computing model for complex intelligent systems," in *Proceedings of the 20th Conference on Computer Science and Intelligence Systems (FedCSIS 2025) Kraków, Poland, September 14-17, 2025*, ser. Annals of Computer Science and Information Systems, M. Bolanowski, M. Ganzha, L. A. Maciaszek, M. Paprzycki, and D. Ślęzak, Eds., 2025, pp. 59–72. [Online]. Available: doi.org/10.15439/2025F6355

[11] F. P. Brooks, Jr., *The Mythical Man-Month: Essays on Software Engineering, Anniversary Edition*. Addison-Wesley, 1995.

[12] M. Jamshidi, Ed., *Systems of Systems Engineering. Principles and Applications*. CRC Press, 2017.

[13] J. J. Kacprzyk, L. Kovács, and I. Rudas, "Some remarks on a way from systems through large scale systems and complex systems to systems of systems," in *From Large-Scale Systems to System of Systems Engineering*, ser. Studies in Systems, Decision and Control, M. Saif, B. Shafai, and J. Kacprzyk, Eds. Springer, 2026, pp. 1–13.

[14] C. Gershenson and F. Heylighen, "How can we think the complex?" in *Managing Organizational Complexity: Philosophy, Theory and Application*, K. Richardson, Ed. Information Age Publishing, 2005, pp. 47–61.

[15] Y. Bar-Yam, *Dynamics of Complex Systems. Studies in Nonlinearity*. Reading, Massachusetts: Addison-Wesley, 1997.

[16] M. Gell-Mann, *The Quark and the Jaguar - Adventures in the Simple and the Complex*. London: Brown & Co., 1994.

[17] J. H. Holland, *Signals and Boundaries. Building Blocks for Complex Adaptive Systems*. Cambridge, MA: The MIT Press, 2014. [Online]. Available: doi.org/10.7551/mitpress/9412.001.0001

[18] W. Pedrycz, *Granular Computing. Analysis and Design of Intelligent Systems*. CRC Press, Taylor & Francis, 2013. [Online]. Available: doi.org/10.1201/9781315216737

[19] W. Pedrycz, A. Skowron, and V. Kreinovich, Eds., *Handbook of Granular Computing*. Hoboken, NJ: John Wiley & Sons, 2008. [Online]. Available: doi.org/10.1002/9780470724163

[20] S. J. Russell and P. Norvig, *Artificial Intelligence A Modern Approach*. Upper Saddle River, NJ: Pearson Education, Inc., 2021, 4th edition.

[21] M. A. Niazi and A. Hussain, *Cognitive Agent-based Computing-I. A Unified Framework for Modeling Complex Adaptive Systems Using Agent-based & Complex Network-based Methods*, ser. SpringerBriefs in Cognitive Computation. Heidelberg: Springer, 2013.

[22] G. Bel-Enguix and M. D. Jiménez-López, Eds., *Language as a Complex System: Interdisciplinary Approaches*. Wiley-Blackwell, 2009.

[23] P. Mercorelli, W. Zhang, H. Nemati, and Y. Zhang, Eds., *Modeling, Identification, and Control for Cyber- Physical Systems Towards Industry 4.0*, ser. Emerging Methodologies and Applications in Modelling. Academic Press, 2024.

[24] C. G. Kirwan and F. Zhiyong, *Smart Cities and Artificial Intelligence. Convergent Systems for Planning, Design, and Operation*. Elsevier, 2025.

[25] X. xiao Long, Q. Zhao, K. Zhang, Z. Zhang, D. Wang, Y. Liu, Z. Shu, Y. Lu, S. Wang, X. Wei, W. Li, W. Yin, Y. Yao, J. Pan, Q. Shen, R. Yang, X. Cao, and Q. Dai, "A survey: Learning embodied intelligence from physical simulators and world models," *ArXiv*, vol. abs/2507.00917, 2025.

[26] R. Mariani, F. Rossi, R. Cucchiara, M. Pavone, B. Simkin, A. Koene, and J. Papenbrock, "Trustworthy AI - part III," *Computer*, vol. 57, no. 3, pp. 22–24, 2024. [Online]. Available: https://doi.org/10.1109/MC.2024.3359766

[27] J. Ji, T. Qiu, B. Chen, J. Zhou, B. Zhang, D. Hong, H. Lou, K. Wang, Y. Duan, Z. He, L. Vierling, Z. Zhang, F. Zeng, J. Dai, X. Pan, H. Xu, A. O'Gara, K. Ng, B. Tse, J. Fu, S. McAleer, Y. Wang, M. Yang, Y. Liu, Y. Wang, S. Zhu, Y. Guo, Y. Yang, and W. Gao, "AI alignment: A contemporary survey," *ACM Comput. Surv.*, vol. 58, no. 5, pp. 132:1–132:38, 2026. [Online]. Available: https://doi.org/10.1145/3770749

[28] S. Thurner, R. Hanel, and P. Klimek, *Introduction to the Theory of Complex Systems*. Oxford University Press, 2018.

[29] W. B. Rouse, *Modeling and Visualization of Complex Systems and Enterprises: Explorations of Physical, Human, Economic, and Social Phenomena*. Hoboken, NJ: John Wiley & Sons, 2015.

[30] C. L. Ortitz Jr., "Why we need a physically embodied Turing test and what it might look like," *AI Magazine*, vol. 37, pp. 55–62, 2016. [Online]. Available: doi.org/10.1609/aimag.v37i1.2645

[31] J. E. Delanois, S. Joshi, R. Golden, T. Nick, and M. Bazhenov, "Context is all you need," *ArXiv*, vol. abs/2604.04364, 2026.

[32] D. Deutsch, E. Ekert, and R. Lupacchini, "Machines, logic and quantum physics," *Neural Computation*, vol. 6, pp. 265–283, 2000. [Online]. Available: doi.org/10.2307/421056

[33] A. Campagner, D. Ciucci, and V. Dorigatti, "Uncertainty representation in dynamical systems using rough set theory," *Theoretical of Computer Science*, vol. 908, pp. 28–42, 2022.

[34] L. Dong, R. Wang, and D. Chen, "Incremental feature selection with fuzzy rough sets for dynamic data sets," *Fuzzy Sets Syst.*, vol. 467, p. 108503, 2023.

[35] M. Wolski, "Rough sets in terms of discrete dynamical systems," in *Rough Sets and Current Trends in Computing - 7th International Conference, RSCTC 2010, Warsaw, Poland, June 28-30,2010. Proceedings*, ser. Lecture Notes in Computer Science, M. S. Szczuka, M. Kryszkiewicz, S. Ramanna, R. Jensen, and Q. Hu, Eds., vol. 6086. Springer, 2010, pp. 237–246.

[36] N. Japkowicz and M. Shah, *Evaluating Learning Algorithms: A Classification Perspective*. New York, NY: Cambridge University Press, 2011.

[37] D. Guégan and B. K. Hassani, *Risk Measurement. From Quantitative Measures to Management Decisions*. Cham: Springer Nature, 2019.

[38] S. Jansen, *Machine Learning for Algorithmic Trading*. Birmingham - Mumbai: Packt, 2020.

[39] N. Brown, Ed., *Artificial Intelligence in Drug Discovery*, ser. Drug Discovery Series. London: Royal Society of Chemistry, 2021, vol. 75.

[40] J. Xu, *Biological Computing*. Singapore: Springer Nature, 2025. [Online]. Available: doi.org/10.1007/978-981-96-3870-3_12

[41] J. Rissanen, "Minimum-description-length principle," in *Encyclopedia of Statistical Sciences*, S. Kotz and N. Johnson, Eds. New York, NY: John Wiley & Sons, 1985, pp. 523–527.

[42] Z. Pawlak and A. Skowron, "Rough sets and boolean reasoning," *Information Sciences*, vol. 177, no. 1, pp. 41–73, 2007. [Online]. Available: doi.org/10.1016/j.ins.2006.06.006

[43] A. Skowron and J. Stepaniuk, "Tolerance approximation spaces," *Fundam. Informaticae*, vol. 27, no. 2/3, pp. 245–253, 1996. [Online]. Available: doi.org/10.3233/FI-1996-272311

[44] H. S. Nguyen, "Approximate boolean reasoning: Foundations and applications in data mining," *Transactions on Rough Sets V: Journal Subline*, vol. 5, pp. 344–523, 2006.
[45] H. S. Nguyen and A. Skowron, "Rough sets: From rudiments to challenges," in *Rough Sets and Intelligent Systems. Professor Zdzislaw Pawlak in Memoriam*, ser. Series Intelligent Systems Reference Library, A. Skowron and Z. Suraj, Eds. Springer, 2013, pp. 75–173.
[46] Y. Yao and B. Yao, "Covering based rough set approximations," *Information Sciences*, vol. 200, pp. 91–107, 2012. [Online]. Available: doi.org/10.1016/j.ins.2012.02.065
[47] L. Ma and M. Li, "Covering rough set models, fuzzy rough set models and soft rough set models induced by covering similarity," *Information Sciences*, vol. 689, p. 121520, 2025. [Online]. Available: doi.org/10.1016/j.ins.2024.121520
[48] X. Ma, Y. Yao, and J. Zhan, "A survey on fuzzy β-covering rough set models, measurement systems, and applications: Advances and prospects from 2016–2026," *Fuzzy Sets and Systems*, vol. 536, p. 109894, 2026. [Online]. Available: doi.org/10.1016/j.fss.2026.109894
[49] T. Hastie, R. Tibshirani, and J. H. Friedman, *The Elements of Statistical Learning: Data Mining, Inference, and Prediction*. Heidelberg: Springer-Verlag, 2001. [Online]. Available: doi.org/10.1007/978-0-387-84858-7
[50] K. Liu, X. Yang, W. Ding, H. Ju, T. Li, J. Wang, and T. Yin, "A survey on rough feature selection: Recent advances and challenges," *IEEE CAA Journal of Automatica Sinica*, vol. 13, no. 3, pp. 521–542, 2026. [Online]. Available: https://doi.org/10.1109/JAS.2025.125231
[51] J. Chen and P. Zhu, "Discernibility matrix-based feature selection approaches with fuzzy dominance-based neighborhood rough sets," *Fuzzy Sets and Systems*, vol. 513, p. 109384, 2025. [Online]. Available: https://doi.org/10.1016/j.fss.2025.109384
[52] P. Stone, *Layered Learning in Multi-Agent Systems: A Winning Approach to Robotic Soccer*. Cambridge, MA: The MIT Press, 2000.
[53] J. Bazan, "Hierarchical classifiers for complex spatio-temporal concepts," in *Transactions on Rough Sets IX: Journal Subline*, ser. Lecture Notes in Computer Science, J. F. Peters, A. Skowron, and H. Rybiński, Eds. Heidelberg: Springer, 2008, vol. 5390, pp. 474–750.
[54] M. A. R. Ferreira, Ed., *Modeling Spatio-Temporal Data: Markov Random Fields, Objective Bayes, and Multiscale Models*. Chapman and Hall/CRC, 2024.
[55] C. Gershenson, "Guiding the self-organization of cyber-physical systems," *Frontiers Robotics AI*, vol. 7, pp. 1–13, 2020. [Online]. Available: https://doi.org/10.3389/frobt.2020.00041
[56] L. Valiant, *Probably Approximately Correct. Nature's Algorithms for Learning and Prospering in a Complex World*. New York, NY: Basic Books, A Member of the Perseus Books Group, 2013.
[57] D. Matthews, A. Spielberg, D. Rus, S. Kriegman, and J. Bongard, "Efficient automatic design of robots," *Proceedings of the National Academy of Sciences (PNAS)*, vol. 120, pp. 1–7, 2023. [Online]. Available: doi.org/10.1073/pnas.2305180120
[58] V. Dolzhenko, *Algorithmic Trading Systems and Strategies: A New Approach: Design, Build, and Maintain an Effective Strategy Search Mechanism*. New York, NY: Apress Media, 2024. [Online]. Available: doi.org/10.1007/979-8-8688-0357-4
[59] R. Gupta, S. Tiwari, and P. Chaudhary, *Generative AI: Techniques, Models and Applications*. Cham: Springer Nature, 2025. [Online]. Available: doi.org/10.1007/978-3-031-82062-5
[60] S. Gerrish, *How Smart Machines Think*. Cambridge, MA: MIT Press, 2018. [Online]. Available: doi.org/10.7551/mitpress/11440.001.0001
[61] L. A. Zadeh, "Fuzzy Sets," *Information and Control*, vol. 8, pp. 338–353, 1965.
[62] ——, "Toward a theory of fuzzy information granulation and its centrality in human reasoning and fuzzy logic," *Fuzzy Sets and Systems*, vol. 90, pp. 111–127, 1997. [Online]. Available: doi.org/10.1016/S0165-0114(97)00077-8
[63] ——, "Toward a Theory of Fuzzy Information Granulation and its Centrality in Human Reasoning and Fuzzy Logic," *Fuzzy Sets and Systems*, vol. 90, no. 2, pp. 111–127, 1997.
[64] S. Pal, "Rough Set and Deep Learning: Some Concepts," *Academia Letters*, vol. , no. Article 1849, pp. 1–6, July 2021.
[65] Z. Chen and B. Liu, *Lifelong Machine Learning*, ser. Synthesis Lectures on Artificial Intelligence and Machine Learning. Cham: Springer Nature, 2018, vol. 38.
[66] L. Jiao, X. Song, C. You, X. Liu, L. Li, P. Chen, X. Tang, Z. Feng, F. Liu, Y. Guo, S. Yang, Y. Li, X. Zhang, W. Ma, S. Wang, J. Bai, and B. Hou, "AI meets physics: a comprehensive survey," *Artiicial Intelligence Review*, vol. 57, no. 9, p. 256, 2024.
[67] H. Kuai, J. X. Huang, X. Tao, G. Pasi, Y. Yao, J. Liu, and N. Zhong, "Web intelligence (wi) 3.0: in search of a better-connected world to create a future intelligent society," *Artificial Intelligence Review*, vol. 58, p. 265, 2025.