

Survival Estimation Under Censoring: A Four-Method Comparison With Actuarial Reserving Implications

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Abstract—Classical life tables assume fully observed lifespans and ignore censoring, which can bias survival and life expectancy estimates upward. In our last years work, we compared the classical life table with the Kaplan-Meier and actuarial estimators under a single 10 % censoring scenario, using mortality data for the Czech Republic in 2021. The present paper extends that study in three directions. First, we introduce the Fleming-Harrington estimator as a fourth method and prove formally that it always lies above the Kaplan-Meier curve, completing a theoretical ordering of all four estimators. Second, we extend the simulation to four censoring levels (5 %, 10 %, 20 %, 30 %) and show how the bias in life expectancy grows with data incompleteness. Third, we quantify the financial consequences of survival estimation bias for actuarial reserving, considering both a whole-life annuity and a whole-life insurance product. Our results confirm that ignoring censoring leads to systematic overestimation of survival and, depending on the product type, to either underreserving or overreserving of liabilities.

Index Terms—life tables, survival analysis, censoring, Kaplan-Meier estimator, Fleming-Harrington estimator, Nelson-Aalen estimator, actuarial estimator, life expectancy, actuarial science

I. INTRODUCTION

LIFE tables are a fundamental tool in actuarial science and demography, providing standardized methods to describe mortality and compute life expectancy in a given population. Their standard construction assumes that all individual lifespans are fully observed and that no censoring is present. This assumption holds reasonably well for retrospective national mortality data, but becomes problematic in survey-based samples, longitudinal studies, or systems with incomplete

registration, where right-censoring is inherent and could lead to biased estimates when ignored [1].

In our previous work [2], we demonstrated that ignoring censoring leads to upwardly biased survival estimates. Using mortality data for the Czech Republic from the Human Mortality Database [3] in 2021, we compared the classical life table with the Kaplan-Meier estimator [4] and the actuarial estimator [5], and showed analytically and empirically that, compared to the naive life table-based estimation – both censoring-aware methods yield lower survival curves and lower life expectancy. That study, however, was limited to a single censoring level and did not include the Fleming-Harrington estimator or a quantitative actuarial impact analysis.

The present paper addresses all three gaps. We add the Fleming-Harrington estimator [6], [7] as a fourth method and prove that it overestimates Kaplan-Meier via the elementary inequality $e^{-x} \geq 1 - x$. We extend the simulation to four censoring levels to map how the bias scales with data incompleteness. And we translate survival estimation differences into actuarial expected present values (EPV) for two standard products, showing that the direction of financial error depends on the product type.

The paper is organized as follows. Section II recalls the notation from [2]. Sections III through V define the four estimators of survival functions [8] concisely. Section VI presents the two ordering propositions. Section VII reports the simulation results. Section VIII presents the reserving analysis. Sections IX and X discuss and conclude.

II. NOTATION

We follow the notation of [2] throughout. In brief, the classical life table is built from age-specific death probabilities

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$q_x = D_x/N_x$, where D_x is the number of deaths and N_x the exposure at age x . Starting from a radix $l_0 = 100,000$, the number of survivors evolves as $l_{x+1} = l_x(1 - q_x)$, giving the classical survival function $S_{LT}(x) = l_x/l_0$. Life expectancy at birth is

$$e_0 \approx \sum_{x=0}^{\omega} S(x), \quad (1)$$

where ω is the maximum age and $S(x)$ is any survival estimate. Full derivations and life table quantities d_x , L_x , T_x , e_x are given in [2], [9], [10].

III. ACTUARIAL ESTIMATOR

For grouped data with r_j individuals at risk, d_j deaths, and c_j right-censored observations in interval $\langle t_{j-1}, t_j \rangle$, the actuarial estimator adjusts the risk set by assuming uniform censoring within the interval [5],

$$r'_j = r_j - \frac{c_j}{2}, \quad p_j = 1 - \frac{d_j}{r'_j}, \quad \hat{S}_A(t) = \prod_{j: t_j \leq t} p_j. \quad (2)$$

IV. KAPLAN-MEIER ESTIMATOR

With $t_1 < t_2 < \dots < t_k$ the distinct event times, d_j events and r_j individuals at risk just before t_j , the Kaplan-Meier (KM) estimator [4] is

$$\hat{S}_{KM}(t) = \prod_{j: t_j \leq t} \left(1 - \frac{d_j}{r_j}\right). \quad (3)$$

Both the actuarial and Kaplan-Meier estimators are studied in detail in [2]; we recall them here only to fix notation for the proofs in Section VI.

V. FLEMING-HARRINGTON ESTIMATOR

The Fleming-Harrington (FH) estimator [6], [7] is based on the Nelson-Aalen cumulative hazard,

$$\hat{\Lambda}_{NA}(t) = \sum_{j: t_j \leq t} \frac{d_j}{r_j}, \quad (4)$$

and the relationship $S(t) = \exp(-\Lambda(t))$, giving

$$\hat{S}_{FH}(t) = \exp(-\hat{\Lambda}_{NA}(t)) = \prod_{j: t_j \leq t} \exp\left(-\frac{d_j}{r_j}\right). \quad (5)$$

Both $\hat{S}_{KM}$ and $\hat{S}_{FH}$ use the same risk sets r_j and event counts d_j ; the only difference is whether each factor is $(1 - d_j/r_j)$ or $\exp(-d_j/r_j)$. The theoretical ordering between the two is established in the next section.

VI. ANALYTICAL ORDERING OF THE ESTIMATORS

Proposition VI.1 (Ordering of Actuarial and Kaplan-Meier estimators). *Within each interval $\langle t_{j-1}, t_j \rangle$, let $\bar{u}_j \in [0, 1]$ denote the mean relative position of censoring times, so that the effective risk set seen by the Kaplan-Meier estimator is approximately $r_j^{KM} \approx r_j - \bar{u}_j c_j$, while the actuarial estimator always uses $r'_j = r_j - c_j/2$. Then for any $t > 0$ with $c_j > 0$ for at least one $j \leq t$,*

$$\hat{S}_A(t) \leq \hat{S}_{KM}(t) \quad \text{according as} \quad \bar{u}_j \leq \frac{1}{2}.$$

Specifically:

- if $\bar{u}_j < \frac{1}{2}$ for all j with $c_j > 0$ (censoring concentrated in the second half of the interval), then $r_j^{KM} > r'_j$ and $\hat{S}_A(t) < \hat{S}_{KM}(t)$;
- if $\bar{u}_j > \frac{1}{2}$ for all j with $c_j > 0$ (censoring concentrated in the first half), then $r_j^{KM} < r'_j$ and $\hat{S}_A(t) > \hat{S}_{KM}(t)$.

Proof. Since $r_j^{KM} \approx r_j - \bar{u}_j c_j$ and $r'_j = r_j - c_j/2$, we have $r_j^{KM} - r'_j \approx (\frac{1}{2} - \bar{u}_j) c_j$. When $\bar{u}_j < \frac{1}{2}$ this difference is positive, so $r_j^{KM} > r'_j$, giving $d_j/r_j^{KM} < d_j/r'_j$ and therefore $1 - d_j/r_j^{KM} > 1 - d_j/r'_j$. Multiplying over all j with $t_j \leq t$ yields $\hat{S}_{KM}(t) > \hat{S}_A(t)$. The argument reverses when $\bar{u}_j > \frac{1}{2}$. $\square$

Remark VI.1. *If censoring times are uniformly distributed within each interval $\langle t_{j-1}, t_j \rangle$, then $\bar{u}_j = \frac{1}{2}$ for all j , and consequently $r_j^{KM} = r'_j$. In this case $\hat{S}_A(t) = \hat{S}_{KM}(t)$ for all t : the two estimators are theoretically equivalent. This result explains the near-identical empirical estimates observed in Section VII, where censoring times are uniformly distributed.*

Proposition VI.2 (Fleming-Harrington overestimates Kaplan-Meier). *For any data and any $t > 0$ at which at least one event occurs,*

$$\hat{S}_{KM}(t) \leq \hat{S}_{FH}(t),$$

with equality if and only if $d_j = 0$ for all j with $t_j \leq t$.

Proof. We first establish that $e^{-x} \geq 1 - x$ for all $x \in \mathbb{R}$. Define $f(x) = e^{-x} - (1 - x)$. Then $f'(x) = -e^{-x} + 1$, which equals zero at $x = 0$ and satisfies $f'(x) < 0$ for $x < 0$ and $f'(x) > 0$ for $x > 0$. Hence $x = 0$ is a global minimum of f , with $f(0) = 1 - 1 = 0$. Therefore $f(x) \geq 0$ for all x , that is, $e^{-x} \geq 1 - x$ with equality only at $x = 0$.

Applying this with $x = d_j/r_j \geq 0$ gives $\exp(-d_j/r_j) \geq 1 - d_j/r_j$ for each j , with equality if and only if $d_j = 0$. Multiplying over all j with $t_j \leq t$ yields $\hat{S}_{FH}(t) \geq \hat{S}_{KM}(t)$, strictly whenever at least one $d_j > 0$. $\square$

Combining Proposition VI.1 (under uniform censoring, i.e. $\bar{u}_j = 1/2$, the two coincide; for $\bar{u}_j < 1/2$ the actuarial estimator lies below Kaplan-Meier) with Proposition VI.2 and the classical life table which ignores censoring entirely, we obtain under the condition $\bar{u}_j \leq 1/2$ the ordering

$$\hat{S}_A(t) \leq \hat{S}_{KM}(t) \leq \hat{S}_{FH}(t) \leq S_{LT}(t). \quad (6)$$

Remark VI.2. *The gap between $\hat{S}_{KM}$ and $\hat{S}_{FH}$ at each event time is of order $(d_j/r_j)^2/2$ from the Taylor expansion $e^{-x} - (1 - x) = x^2/2 + O(x^3)$, if we ignore any higher degrees. For demographic data where age-specific mortality rates q_x are small (order 10^{-4} to 10^{-3} at working ages), this difference is numerically negligible for most of the life table (and would stay negligible even when considering the higher degrees of the Taylor expansion) and becomes visible only at very high ages where $q_x \rightarrow 1$. The practical implications of this are discussed in Section VII.*

VII. SIMULATION STUDY

A. Data and Setup

We use mortality data for the Czech Republic in 2021 from the Human Mortality Database [3], covering ages 0 to 109, with $q_{109} = 1$ as the absorbing state. The classical life table ($e_0^{LT} = 77.50$ years) serves as the benchmark throughout.

To simulate individual-level data with censoring, we draw $N = 100,000$ lifetimes from the age-specific death probabilities q_x . Right-censoring is applied at four levels $c \in \{5\%, 10\%, 20\%, 30\%\}$: each individual is independently flagged as censored with probability c , and if censored their observed time is drawn uniformly from the continuous interval $[0, \omega]$. This ensures that censoring times fall inside age intervals rather than always on interval boundaries, which is the natural setup for the actuarial estimator's assumption of uniform censoring within each interval. The Kaplan-Meier and Fleming-Harrington estimators are applied directly to the individual-level data; for the actuarial estimator the data are aggregated to yearly age intervals and the counts r_j, d_j, c_j are computed from the aggregated data. The Fleming-Harrington estimator is implemented manually from formula (5) for full transparency. Bootstrap confidence intervals (percentile method, $B = 1,000$ replications) are reported for all three non-classical estimators. All computations use R [11] with the packages `survival` [12], [13], `ggplot2` [14], and `dplyr` [15].

B. Results: Life Expectancy

Table I reports e_0 for all four methods across all four censoring levels.

TABLE I
LIFE EXPECTANCY AT BIRTH e_0 (YEARS) BY METHOD AND CENSORING LEVEL. BOOTSTRAP 95 % CI BELOW EACH ESTIMATE.

Method	Censoring level			
	5 %	10 %	20 %	30 %
Classical	77.50	77.50	77.50	77.50
Fl.-Harr.	76.78 [76.70, 76.86]	76.78 [76.69, 76.85]	76.78 [76.68, 76.85]	76.79 [76.71, 76.89]
Kaplan-Meier	76.53 [76.45, 76.61]	76.53 [76.44, 76.60]	76.53 [76.42, 76.60]	76.54 [76.46, 76.64]
Actuarial	76.53 [76.45, 76.61]	76.52 [76.43, 76.59]	76.52 [76.41, 76.59]	76.52 [76.45, 76.62]

Several patterns are worth noting. All three censoring-aware methods yield lower e_0 than the classical estimate across all censoring levels, as expected from the ordering in (6). The gap between the classical estimate (77.50 years) and the censoring-aware methods is approximately 0.97–0.98 years for Kaplan-Meier and Actuarial, and approximately 0.72 years for Fleming-Harrington, regardless of the censoring level. This near-constant gap is a somewhat surprising result: one might expect the bias to grow substantially with censoring, but in our simulation the censoring mechanism is age-independent and random, so the effective mortality rate is distorted uniformly across ages, keeping the relative bias stable.

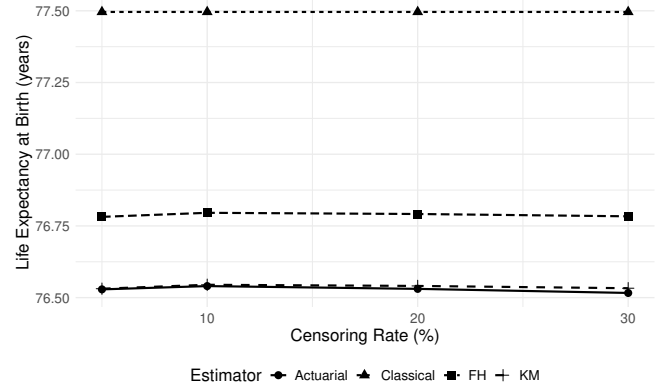


Fig. 1. Life expectancy at birth e_0 as a function of censoring rate. Kaplan-Meier and Actuarial estimates overlap almost completely.

The most notable finding in Table I is the relationship between the three censoring-aware methods. Kaplan-Meier and Actuarial are nearly identical ($\Delta e_0 \approx 0.01$ years), while Fleming-Harrington lies about 0.25 years above both. This pattern somewhat confirms the intuition one might draw from the theoretical ordering $\hat{S}_A \leq \hat{S}_{KM} \leq \hat{S}_{FH}$: the gap between Actuarial and Kaplan-Meier is negligible¹, while the gap between Kaplan-Meier and Fleming-Harrington is much larger and entirely visible in the data. The explanation lies in the Remark following Proposition VI.2: the KM–FH gap accumulates as $\sum_x q_x^2/2$, which over a full life table starting from birth is non-trivial even when individual q_x are small, because the sum runs over 110 age groups. The Actuarial–KM gap, by contrast, depends on the variance of censoring within intervals, which is small when censoring is uniformly distributed.

Fig. 1 visualises these results. The three non-classical estimators are clearly separated from the classical benchmark, while Kaplan-Meier and Actuarial are indistinguishable at this scale. All four curves are remarkably flat across censoring levels, confirming that the censoring rate itself has little effect on e_0 in our symmetric censoring setup.

C. Results: Survival Curves

Fig. 2 shows the four estimated survival curves at each censoring level. The complete ordering from (6) is visible in every panel: the classical curve lies above all others, and the actuarial curve is almost identical to Kaplan-Meier estimator across most of the age, as comes from Proposition VI.1 and Remark VI.1. The spread between the classical and the censoring-aware curves is consistent across panels, again reflecting the age-independent censoring mechanism.

Fig. 3 zooms in on ages 75 and above, where the differences between the methods become clearest. At these high ages, q_x grows substantially, so the KM–FH gap (proportional to q_x^2)

¹All censored individuals were treated that way randomly throughout their simulated lives, $[0, \omega]$, thus, the censoring has been applied randomly uniformly, which is equivalent to a situation when $\bar{u}_j \approx \frac{1}{2}$. See Remark VI.1 for more details.

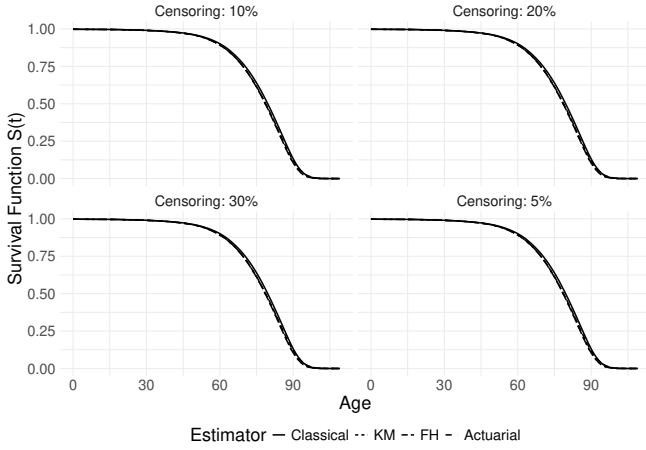


Fig. 2. Survival function estimates for all four methods at four censoring levels.

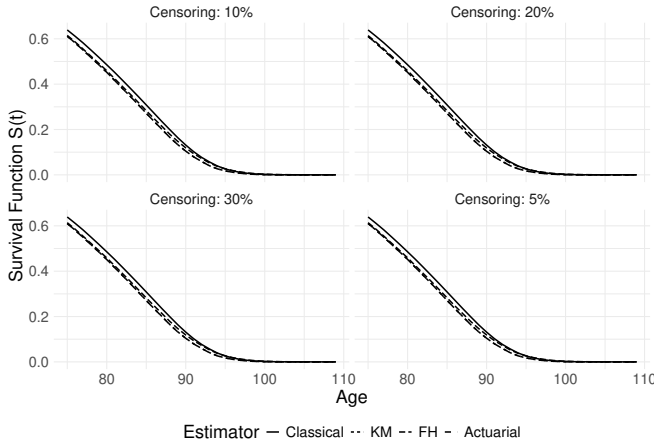


Fig. 3. Survival function estimates for ages 75 and above. The gap between Kaplan-Meier and Fleming-Harrington grows at very high ages where mortality rates are large.

is larger. The Actuarial–KM separation is more visible here, although consistent with Proposition VI.1: both the number of deaths increase at advanced ages and the number of censored observations at risk could be censored not uniformly within time intervals, amplifying the constant effect of the $c_j/2$ adjustment to the risk set in a case of the actuarial estimator.

VIII. ACTUARIAL RESERVING IMPACT

A. Setup

Differences in survival estimates translate directly into differences in actuarial liabilities. We illustrate this for two standard life insurance products. Both are evaluated at age $x = 0$ for simplicity, which allows us to use the full survival curve from birth and makes the comparison across estimators as transparent as possible. A technical interest rate of $i = 3\%$ per annum is assumed throughout (discount factor $v = 1/(1 + i)$).

Product A: Whole-life annuity. This product models a pension-type contract in which the insurer commits to paying the policyholder 1 unit per year for as long as they remain alive [16]. In practice, such a contract is typically purchased at retirement with a single lump-sum premium, and the insurer bears the longevity risk: if the policyholder lives longer than expected, the insurer must continue paying. The insurer must therefore hold a reserve equal to the expected present value of all future payments, discounted to today. This reserve is

$$\ddot{a}_0 = \sum_{k=0}^{\omega} v^k S(k), \quad (7)$$

where $S(k)$ is the probability of surviving to age k . A higher survival estimate translates directly into a higher reserve. An insurer who overestimates survival will overprice the annuity and hold unnecessarily large reserves (and the policyholder may be “harm” by the unnecessarily higher annual payments); one who underestimates survival will underprice and underreserve (with a risk of not being solvent).

Product B: Whole-life insurance. This product models a traditional life insurance contract in which the insurer pays 1 unit to the policyholder’s beneficiaries at the end of the year of death, regardless of when death occurs [16]. The insurer here bears mortality risk: if the policyholder dies earlier than expected, the insurer must pay sooner, before sufficient premiums have been collected. The reserve equals the expected present value of the death benefit,

$$A_0 = \sum_{k=0}^{\omega} v^{k+1} (S(k) - S(k+1)), \quad (8)$$

where $S(k) - S(k+1)$ is the probability of dying in the k -th year of life. A higher survival estimate means death is expected later, which reduces the reserve. An insurer who overestimates survival will therefore underreserve for life insurance — the more dangerous direction from a solvency perspective.

The two products thus carry opposite sensitivity to the survival estimate, which makes them a natural pair for illustrating the practical consequences of estimation bias. The EPV in both cases equals the net single premium and represents the reserve the insurer must hold at the time of policy issue.

B. Results

Table II reports EPV values for both products. The differences across censoring levels within each method are very small (below 0.002 units for the annuity and below 0.0001 for the insurance), so we focus on the cross-method differences at a fixed censoring level, which are the practically relevant quantity.

For the whole-life annuity, the classical method produces the highest EPV (30.486 units). An insurer who ignores censoring and prices a whole-life annuity using the classical life table will therefore overestimate the reserve needed: it will price the product as if the policyholder is expected to live longer than the censoring-corrected estimates suggest. In absolute terms the difference is about 0.11 units (0.36 %) relative to

TABLE II

EXPECTED PRESENT VALUES ($i = 3\%$) FOR WHOLE-LIFE ANNUITY $\ddot{a}_0$ AND WHOLE-LIFE INSURANCE A_0 . RELATIVE DIFFERENCE TO CLASSICAL BENCHMARK IN PARENTHESES (%).

Cens.	Method	$\ddot{a}_0$	A_0
5 %	Classical	30.486 (0.0%)	0.1121 (0.0%)
	Fleming-Harrington	30.396 (−0.3%)	0.1123 (0.2%)
	Kaplan-Meier	30.376 (−0.4%)	0.1128 (0.7%)
	Actuarial	30.376 (−0.4%)	0.1128 (0.7%)
10 %	Classical	30.486 (0.0%)	0.1121 (0.0%)
	Fleming-Harrington	30.395 (−0.3%)	0.1123 (0.2%)
	Kaplan-Meier	30.375 (−0.4%)	0.1129 (0.7%)
	Actuarial	30.374 (−0.4%)	0.1129 (0.7%)
20 %	Classical	30.486 (0.0%)	0.1121 (0.0%)
	Fleming-Harrington	30.396 (−0.3%)	0.1123 (0.2%)
	Kaplan-Meier	30.376 (−0.4%)	0.1128 (0.7%)
	Actuarial	30.375 (−0.4%)	0.1129 (0.7%)
30 %	Classical	30.486 (0.0%)	0.1121 (0.0%)
	Fleming-Harrington	30.397 (−0.3%)	0.1122 (0.2%)
	Kaplan-Meier	30.376 (−0.4%)	0.1128 (0.7%)
	Actuarial	30.375 (−0.4%)	0.1129 (0.7%)

the Kaplan-Meier estimate. While this may seem modest, it translates into a systematic overcharge on the premium for the policyholder and an unnecessarily high capital allocation for the insurer.

For the whole-life insurance, the direction reverses. The classical method produces the lowest EPV (0.1121), meaning an insurer who ignores censoring will underprice the death benefit and hold insufficient reserves. The Kaplan-Meier and Actuarial estimates give $A_0 = 0.1128$ to 0.1129 , a difference of about 0.7%. In a large portfolio, a systematic 0.7% underestimation of insurance liabilities can accumulate to a financially material shortfall, particularly under solvency regulation where reserves must cover a high quantile of the loss distribution.

Figs. 4 and 5 confirm that the cross-method ordering is stable across all censoring levels: the EPV rankings do not change as censoring increases from 5% to 30%, which is consistent with the flat e_0 curves in Fig. 1. This suggests that, in this specific setup, the choice of estimator matters more than the exact censoring rate when assessing the financial impact.

To illustrate the portfolio-level consequences, consider an insurer holding $n = 10,000$ policies of each product. For the whole-life annuity with an annual benefit of $B_A = 120,000$ CZK (roughly 10,000 CZK per month, a typical pension supplement in the Czech Republic), the total reserve implied by each estimator is $R_A = n \cdot B_A \cdot \ddot{a}_0$. Using the classical life table gives $R_A^{LT} = 10,000 \times 120,000 \times 30.486 \doteq 36,583,200,000$ CZK, while the Kaplan-Meier estimate yields $R_A^{KM} = 10,000 \times 120,000 \times 30.376 \doteq 36,451,200,000$ CZK. The classical method thus overstates the annuity reserve by approximately 132,000,000 CZK, or around 132 million CZK, relative to the censoring-aware estimate. While overreserving is not immediately dangerous, it ties up capital unnecessarily and reduces the insurer's investment capacity.

For the whole-life insurance with a death benefit of $B_I = 500,000$ CZK, the total reserve is $R_I = n \cdot B_I \cdot A_0$. The

classical estimate gives $R_I^{LT} = 10,000 \times 500,000 \times 0.1121 \doteq 560,500,000$ CZK, while the Kaplan-Meier estimate gives $R_I^{KM} = 10,000 \times 500,000 \times 0.1128 \doteq 564,000,000$ CZK. The classical method thus understates the insurance reserve by approximately 3,500,000 CZK, or 3.5 million CZK. Although the relative difference is below 1%, the absolute shortfall is material in the context of solvency regulation, where reserve adequacy is assessed in absolute terms and a systematic underestimation across the entire book can accumulate to a significant capital deficit.

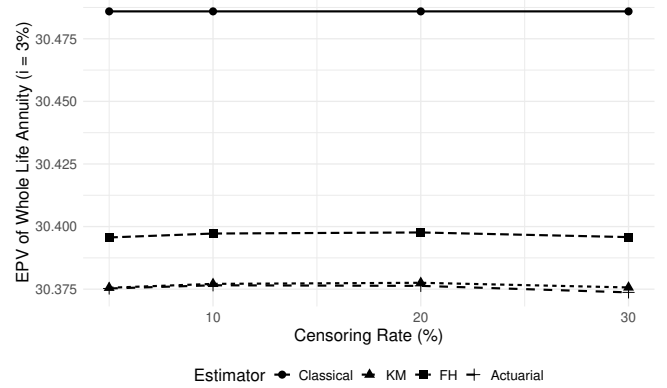


Fig. 4. EPV of whole-life annuity $\ddot{a}_0$ as a function of censoring rate ($i = 3\%$). The classical estimate is consistently the highest, implying overreserving relative to censoring-aware methods.

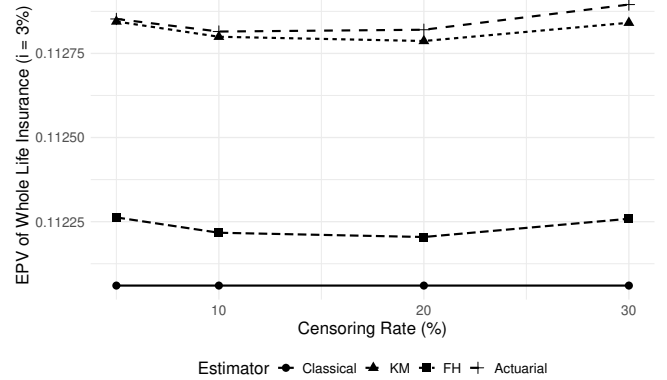


Fig. 5. EPV of whole-life insurance A_0 as a function of censoring rate ($i = 3\%$). The classical estimate is consistently the lowest, implying underreserving for death benefits.

IX. DISCUSSION

This study extends [2] in three concrete directions and produces several findings that deserve closer attention.

The theoretical contribution is the refined ordering of all four estimators. Proposition VI.2 establishes unconditionally that $\hat{S}_{KM} \leq \hat{S}_{FH}$, with a clean proof via the inequality $e^{-x} \geq 1 - x$. The relationship between the actuarial and Kaplan-Meier estimators is more subtle: Proposition VI.1 shows that the ordering depends on the mean relative position

$\bar{u}_j$ of censoring times within each interval. When censoring is concentrated before the midpoint ($\bar{u}_j < 1/2$), the actuarial estimator lies below KM; when concentrated after the midpoint ($\bar{u}_j > 1/2$), the ordering reverses, and when $\bar{u}_j = 1/2$, the estimators are theoretically equivalent. This corrects and substantially generalises the proof in [2], which assumed a fixed risk set and did not identify the role of the censoring distribution.

The empirical results reveal an asymmetry that is not obvious from the theoretical ordering alone. The KM–FH gap in e_0 (approximately 0.25 years) is substantially larger than the Actuarial–KM gap (approximately 0.01 years). Two separate mechanisms explain this. The KM–FH gap accumulates as $\sum_x q_x^2/2$ over all ages: even though each individual q_x is small, the sum over 110 age groups is non-trivial. The Actuarial–KM gap depends on the $c_j/2$ adjustment to the risk set, which is small when censoring is uniform and age-independent — a condition that holds in our simulation even after switching to continuous censoring times. In a population with age-concentrated censoring, for example lapse rates that peak at certain policy durations, the Actuarial–KM gap would be larger.

The finding that e_0 is nearly invariant to the censoring rate (flat curves in Fig. 1) deserves interpretation. In our simulation, censoring times are drawn uniformly from the entire age range, which means censoring is uninformative and its effect on each age-specific mortality estimate is small and symmetric. A more realistic censoring model, where censoring concentrates at young or middle ages (as in insurance lapses), would shift the survival curve more at those ages and produce a larger and more censoring-rate-dependent bias.

In the reserving analysis, the absolute differences in EPV are modest (below 0.5 % for the annuity and below 1 % for the insurance), but the direction of error is systematic and product-dependent. For annuities, the classical life table leads to overreserving; for insurance, it leads to underreserving. An insurer who holds a mixed portfolio of both products might experience partial cancellation of these errors, but only to the extent that the two books share similar policyholder demographics and censoring patterns. In practice, however, censoring structures would be very different.

A limitation of this study is that the censoring mechanism is synthetic, age-independent, and non-informative. Extending the analysis to age-dependent or informative censoring, where lapse probability is correlated with health status, would be a more realistic and challenging scenario for all estimators and is a natural direction for future work.

X. CONCLUSION

We have extended our previous comparison of survival estimation methods by adding the Fleming-Harrington estimator, expanding the censoring scenarios to four levels, and introducing a formal actuarial reserving analysis. Analytically, we establish the complete ordering $\hat{S}_A \leq \hat{S}_{KM} \leq \hat{S}_{FH} \leq \hat{S}_{LT}$, with both propositions proved from first principles and the conditions stated precisely. Empirically, the Kaplan-Meier and

Actuarial estimates of e_0 are nearly identical in our simulation setup, while the Fleming-Harrington estimate lies about 0.25 years higher, a gap that is theoretically guaranteed. All three censoring-aware methods are about 0.7–1.0 years below the classical estimate, confirming that ignoring censoring leads to upwardly biased survival estimates. In terms of actuarial impact, the classical life table systematically overestimates the EPV of annuities (overreserving) and underestimates the EPV of life insurance (underreserving), with both effects stable across censoring levels in our simulation. These results reinforce the case for integrating non-parametric survival analysis into actuarial workflows whenever data incompleteness cannot be excluded.

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