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Kaprekar's transformations. Part I – theoretical discussion

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***Abstract*—The paper is devoted to discussion of the minimal cycles of the so called Kaprekar's transformations and some of its generalizations. The considered transformations are the self-maps of the sets of natural numbers possessing n digits in their decimal expansions. In the paper there are introduced several new characteristics of such maps, among others, the ones connected with the Sharkovsky's theorem and with the Erdős-Szekeres theorem concerning the monotonic subsequences. Because of the size the study is divided into two parts. Part I includes the considerations of strictly theoretical nature resulting from the definition of Kaprekar's transformations. We find here all the minimal orbits of Kaprekar's transformations T_n, for $n = 3, ..., 7$. Moreover, we define many different generalizations of the Kaprekar's transformations and we discuss their minimal orbits for the selected cases. In Part II (ibidem), which is a continuation of the current paper, the theoretical discussion will be supported by the numerical observations. For example, we notice there that each fixed point, familiar to us, of any Kaprekar's transformation generates an infinite sequence of fixed points of the other Kaprekar's transformations. The observed facts concern also several generalizations of the Kaprekar's transformations defined in Part I.**

I. Introduction

SUBJECT concerning the form, description and coexistence of orbits of the given map $F\colon X \to X$ became a chart-topping object of research after popularization of the Sharkovsky's theorem ([1], [8], [9], [10], [23], [27], [28]). We shall recall it to the Readers.

Let $\mathbb{N}$ denote the set of all positive integers. The following ordering of elements of $\mathbb{N}$ is called the Sharkovsky's ordering of $\mathbb{N}$:

$$\begin{gathered} 3, 5, 7, 9, \ldots, 2\cdot 3, 2\cdot 5, 2\cdot 7, 2\cdot 9, \ldots, \\ 2^2\cdot 3, 2^2\cdot 5, 2^2\cdot 7, 2^2\cdot 9, \ldots \\ 2^k\cdot 3, 2^k\cdot 5, 2^k\cdot 7, 2^k\cdot 9, \ldots, 2^4, 2^3, 2^2, 2, 1. \end{gathered} \tag{1}$$

Sharkovsky's theorem. *The following facts hold:*

(a) *If $f\colon [0,1] \to [0,1]$ is a continuous map then there exists $n = n(f) \in \mathbb{N} \cup \{2^\infty\} \cup \{0\}$ such that the set $Per(f)$ of periods of all periodic orbits of f is equal to the set of all $m \in \mathbb{N}$ located on the right side of n in the Sharkovsky's order (if $n = 2^\infty$ then, by definition, $Per(f) = \{2^k\colon k = 0, 1, 2, \ldots\}$, whereas, if $n = 0$ then $Per(f) := \mathbb{N}$).*

(b) *If $n \in \mathbb{N} \cup \{2^\infty\} \cup \{0\}$ then there exists a continuous map $f\colon [0,1] \to [0,1]$ such that the set $Per(f)$ is equal to the set of all $m \in \mathbb{N}$ located on the right side of n in the Sharkovsky's order and for two selected cases, $n = 2^\infty$ and $n = 0$, the set $Per(f)$ is equal to the one defined above.*

In the subject-matter referring to the Sharkovsky's theorem we know a lot at the moment and many facts have been also till now discovered, like for example the description of periodic orbits of triod (see [2]), the generalizations of Sharkovsky's theorem for hereditarily decomposable chainable continua (see [22], [25], [26]) and the new order for periodic orbits of interval maps (see [5] and references therein). Another important fact (which we intend to discuss in this study as well) concerns not only the periods of a given map but also the so called orbit type. It was at first defined by S. Baldwin in [3] for maps of an interval (see also [24] and references therein) and next extended by others (for example in [4] for the maps of a circle and in [21] for the groups and the groups of graphs). We will use here the following definition [1]. If $f\colon X \to X$, where $X \subset \mathbb{R}$ has n-elements (minimal) orbit $\{x_0, f(x_0), \ldots, f^{n-1}(x_0)\}$, where f^k denotes the $k-$times composition of f, then this orbit induces a cyclic permutation of order n, called the orbit type. More precisely, if the points of this orbit are indexed in increasing order $x_1 < x_2 < \ldots < x_n$, then the respective orbit type p is defined by $p(k) = j$ whenever $f(x_k) = x_j$. In other words, if $x_0 = x_{k_1}$, $f(x_0) = x_{k_2}, \ldots, f^{(n-1)}(x_0) = x_{k_n}$, then the orbit type p is equal to $(k_1, k_2, \ldots, k_n)$. We note that there exists $(n-1)!$ orbit types of order n.

We say that the orbit type guarantees a period-3 point if any continuous function with an orbit of that type possesses a three-element orbit. Eric Lundberg proved in paper [19] that

$$\lim_{n\to\infty} \frac{\gamma_n}{(n-1)!} = 1,$$

where γ_n denotes the number of orbit types of order n that guarantees a period-3 point.

Let us emphasize that almost all the above results cannot be transformed so obviously onto many equally interesting cases of maps, even so numerically attractive like the self-maps of the finite sets.

A reason for creating this paper was the information, surprising for the Authors, about the existence of the so called Kaprekar constant [16], [17], which appeared to be,

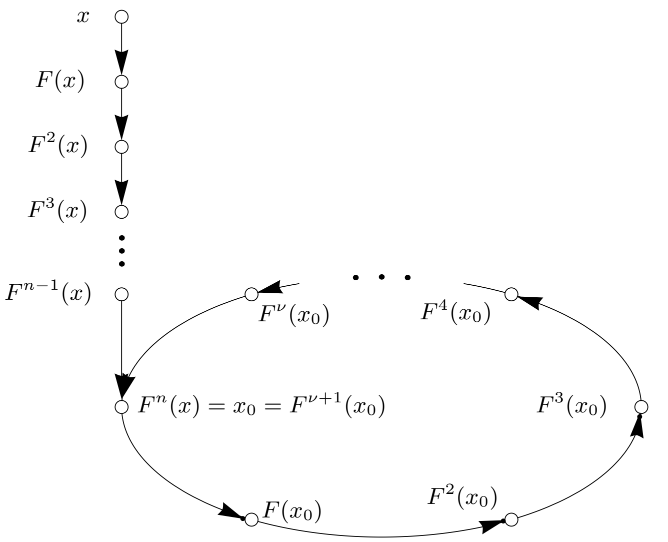

Fig. 1. Graphical illustration of a finite set X and a map $F\colon X \to X$, where $X = \{F^k(x) : k \in \mathbb{N}\}$ for some $x \in X$, possessing one nontrivial and proper orbit

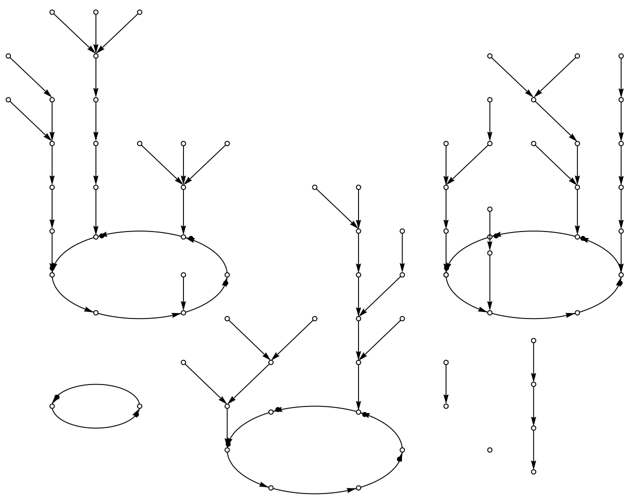

Fig. 2. Graphical illustration of any map $F : X \to X$ operating, where X is a finite set of all indicated circle-points

no more no less, a single element of a single orbit of some map (we will describe this map in Section 2) onto the finite set of all natural numbers with four-digit decimal expansion. Let us notice in this moment that if $F\colon X \to X$ and X is a finite nonempty set then for every $x \in X$ there exists $n \in \mathbb{N}_0$ such that n-th F-iteration of x, i.e. the element $F^n(x)$, belongs to some minimal orbit of F. This means, by definition, that certain subset of X is of the form $\{x_0 = F^{\nu+1}(x_0), F(x_0), F^2(x_0), \ldots, F^\nu(x_0)\}$, where $\nu \in \mathbb{N}_0$. The above facts are illustrated in Figures 1 and 2.

Let us note that in general case there is no connection between values n and ν (more precisely, for any $n, \nu \in \mathbb{N}$, for the set X composed of elements – circles like in Fig.1, we construct a map described in Fig.1 proving that there is no relation between n and ν). However, we should remember that in the case of some specific maps (and even for the families of maps) the relation between n and ν may appear!

In case when F is a bijection on X, that is permutation on X, then every element of set X belongs to some F-orbit (F-orbit is created by elements of each cycle of permutation F). Certainly, if F is not a bijection on X then the situation is also easy to describe, at least from the theoretical point of view, namely the set

$$\mathbb{X} := \bigcap_{k=0}^{card\, X} F^k(X)$$

is a set-theoretical union of all orbits of the map F, and moreover, F restricted to $\mathbb{X}$ is a bijection on $\mathbb{X}$. Set $\mathbb{X}$ is the largest fixed subset of map F, it means if $Y \subset X$ and $F(Y) = Y$, then $Y \subset \mathbb{X}$. Henceforward we will call such set as the maxinvariant subset of F. The only problem in this situation is the actual form of set $\mathbb{X}$? (In Figure 2 the set $\mathbb{X}$ is equal to the union of final single points and all points located on the indicated ellipses.) Of course equally essential, although much more difficult in practice, is the description of all orbits of map F.

In this paper, as the input set X we will take the families containing numbers 0, $10^{k-1} - 1$ and the natural numbers possessing k digits in their decimal expansion, that is

$$X = X(k) = \{0\} \cup \{n \in \mathbb{N}\colon 10^{k-1} - 1 \leq n < 10^k\}$$

for each $k \in \mathbb{N}$. This additional "condition" will enable to reduce determination of the orbits of the so called Kaprekar's transformations $T_k\colon X(k) \to X(k)$ – described in the next section – to solution of some diophantine equations. Although we have learnt about orbits of many maps T_k, this knowledge did not help us unfortunately to answer the basic question: how many orbits do these maps possess in dependence on the value of parameter k for any $k \in \mathbb{N}$? In both parts of our study we are able to answer this question only for values $k \leq 20$.

In Part II of our considerations we will present many various remarks, facts and conjectures which arose basically by observing the numerical results concerning the description of the orbits of maps T_n for $n \leq 20$. We will prove, among others, that the fixed points of these maps generate the infinite sequences of the fixed points of maps $T_{a\,n+b}$, $n \in \mathbb{N}$, for some natural numbers a and b.

Additionally, we have noticed that many from among the maps investigated by us (including the generalizations of the Kaprekar's transformation – we define them in last section – however, with regard to this paper length, we will present the appropriate considerations in a separate paper) preserve the strong Sharkovsky's order (the Sharkovsky's order, respectively). It should be understood in the following way.

Definition 1. *Map $T\colon X \to X$, where X is a finite set, preserves the strong Sharkovsky's order if the elements of the set of cardinalities of all orbits of this map can be ordered* *in the sequence $k_1, k_2, \ldots, k_n$, being the sequence of natural numbers, successive in the sense of order (1).*

Definition 2. *Map $T\colon X \to X$, where X is a finite set, preserves the Sharkovsky's order if the elements of the set of cardinalities of all orbits of this map can be ordered in the sequences $k_1^{(r)}, k_2^{(r)}, \ldots, k_n^{(r)}$, $r = 1, 2, \ldots, s$, successive in the sense of order (1), and the different values of superscript r correspond with the different "numbers of levels" of description (1). More precisely, the first level of description (1) is formed by the numbers*

$$3, 5, 7, 9, 11, \ldots,$$

the second level of description (1) is made by the numbers

$$2\cdot 3, 2\cdot 5, 2\cdot 7, 2\cdot 9, 2\cdot 11, \ldots,$$

the third level of description (1) is created by the numbers

$$4\cdot 3, 4\cdot 5, 4\cdot 7, 4\cdot 9, 4\cdot 11, \ldots, \quad \textit{and so on},$$

and finally "the last level" of description (1) is formed by the numbers

$$\ldots, 2^5, 2^4, 2^3, 2^2, 2, 1.$$

Reason of these definitions is also worth to recall. So, as it is easy to prove, for any one-to-one sequence $k_1, k_2, \ldots, k_n$ of natural numbers there exist the sets X_i, $i = 1, 2, \ldots, n$, (pairwise disjoint and such that $cardX_i = k_i$) and the map $T\colon \bigcup_{i=1}^{n} X_i \to \bigcup_{i=1}^{n} X_i$, for which the sets X_i are the only minimal orbits.

Moreover, we have investigated the minimal cycles of the discussed here maps with regard to the Erdős-Szekeres theorem, as well as to the maximal length of monotonic intervals of the given cycle (see [30]) and, at last, by paying the special attention to the relatively new but extremely dynamic theory of "pattern avoiding permutations" (see [6], [20]).

Let us recall here at least few essential definitions and facts. Let $\mathbf{a} = \{a_i\}_{i=1}^{n}$ be a one-to-one sequence of real numbers. Each subsequence $\mathbf{b}$ of $\mathbf{a}$ having the form $\{a_l, a_{l+1}, ..., a_{l+r}\}$ for some $l, r \in \mathbb{N}_0$, $1 \le l \le l+r \le n$, will be called an interval of $\mathbf{a}$. A subsequence $\mathbf{b}$ of $\mathbf{a}$ is said to be a monotonic interval of $\mathbf{a}$ whenever $\mathbf{b}$ is an interval of $\mathbf{a}$ and, simultaneously, $\mathbf{a}$ is a monotonic sequence. Moreover, we will denote by $l(\mathbf{a}) := n$ the number of elements of $\mathbf{a}$ called as the length of $\mathbf{a}$, by $d(\mathbf{a})$ – the maximal number from among the numbers denoting the lengths of all decreasing subsequences of $\mathbf{a}$ and finally by $i(\mathbf{a})$ – the maximal number from among the numbers denoting the lengths of all increasing subsequences of $\mathbf{a}$.

Erdős-Szekeres' theorem. *Let us suppose that $\mathbf{a}$ is a finite one-to-one sequence of real numbers. Then we have*

$$d(\mathbf{a})\, i(\mathbf{a}) \ge l(\mathbf{a}).$$

The above theorem comes from the joint paper by Erdős and Szekeres concerning the Ramseys problem [12]. Next, Wituła et al. in [30] have discussed whether the given one-to-one sequence $\mathbf{a}$ of all numbers $1, 2, ..., n$ (which means that $\mathbf{a}$ can be identified with the respective permutation on set $\{1, 2, ..., n\}$) contains a monotonic interval $\mathbf{b}$ of length 3. The following fact is, among others, proven there.

Theorem 1. *Let $\mathbf{a} = \{a_i\}_{i=1}^{3n}$ be a permutation on $\{1, 2, ..., 3n\}$ and let $n \ge 4$. If $i(\mathbf{a}) = n$, $d(\mathbf{a}) = 3$, $a_k = 3n$ and $a_l = 1$ for some $k < l$, then $\mathbf{a}$ contains a monotonic interval $\mathbf{b}$ of length 3.*

In the next section of this paper we will present the definition of Kaprekar's transformations T_n and we will formulate the conditions describing the elements of minimal orbits of T_n for $4 \le n \le 7$. In fact, it will be only the necessary conditions, yet they will "reduce" enough the sets of natural numbers containing the maxinvariant subset of the respective Kaprekar's transformation, so that the final calculations will be possible to make even by hand.

II. Kaprekar's transformations

In this section we discuss the Kaprekar's transformations

$$T_n\colon \{0\} \cup \{\alpha\colon 10^{n-1} - 1 \le \alpha < 10^n\} \to \\ \to \{0\} \cup \{\alpha\colon 10^{n-1} - 1 \le \alpha < 10^n\}$$

for every $n \in \mathbb{N}$, defined in the following way. We set $T_n(0) = 0$ and let $\alpha \in \mathbb{N}$ be any n-digit number, the decimal expansion of which is composed of digits $0 \le a_1 \le a_2 \le \ldots \le a_n \le 9$. We take

$$T_n(\alpha) := \sum_{k=1}^{n} (a_k - a_{n-k+1})10^{k-1} = \\ = a_n a_{n-1} \ldots a_1 - a_1 a_2 \ldots a_n.$$

The orbits of operator T_n will be called as the T_n-orbits for every $n \in \mathbb{N}$. Moreover, we will call the k-fold composition of operator T_n, for any $k, n \in \mathbb{N}$, as the T_n-composition. Next, the fixed points of operator T_n, where $n \in \mathbb{N}$, will be called as the Kaprekar's constants of n-th order.

Let us note that Hindu mathematician Dattathreya Ramachandra Kaprekar has started in 1949 in paper [16] the discussion on the, called now, Kaprekar's transformations T_n. The classical Kaprekar's constant, that is number 6174, was also announced in this paper. But only in paper [17] Kaprekar proved that after applying operator T_4 at most 7-times every four-digit number in base 10 leads to the same result, that is $6174 = T_4(6174)$.

Properties of operator T_5, acting on the five-digit integers in bases $r < 13$, were investigated by Charles W. Trigg [29], the mathematician well-known mostly for his great involvement in the issues of recreational mathematics. Next, Klaus E. Eldridge and Seok Sagong in their paper [11] from 1988 described the convergence of $\{T_3^n(x)\}_{n=1}^{\infty}$ for all three-digit numbers x for any base $r \in \mathbb{N}$, $r \ge 2$. They obtained, among others, the following result.

Theorem 2.

a) *$T_3^n(x)$ is convergent (in usual sense) to nontrivial constant (also called the Kaprekar's constant) if and only if*

r is even. The respective Kaprekar's constant is equal to the 3-digit number $\left(\frac{r-2}{2}, r-1, \frac{r}{2}\right)$ in base r.

b) *If r is odd then T_3 possesses (except the trivial orbit) only one two-element orbit consisting of numbers $\left(\frac{r-3}{2}, r-1, \frac{r+1}{2}\right)$ and $\left(\frac{r-1}{2}, r-1, \frac{r-1}{2}\right)$ in base r.*

Papers [7], [15], [18] are also devoted to the discussion on Kaprekar's transformations.

We will present now the descriptions of elements of orbits of maps T_n for values of n equal in turn 5,6,7 and 4. These facts are partly new and originally presented.

Theorem 3. *Every orbit of operator T_5 must contain exclusively the numbers of the form $ABA \times 99$, where $0 \leq B \leq A \leq 9$.*

Proof: For five-digit number n composed only of digits $0 \leq e \leq d \leq c \leq b \leq a \leq 9$ we have

$$\begin{aligned} T_5(n) &= (a-e)(10^4-1)+(b-d)(10^3-10) = \\ &= 99 \times ((a-e)\times 101 + (b-d)\times 10) \\ &= 99 \times ABA, \end{aligned}$$

where $0 \leq B := b-d \leq A := a-e \leq 9$. ■

Corollary 1. *The orbits of operator T_5 can be sought just and only from among the T_5-iteration of the following 54 numbers*

$$\begin{aligned} &101\times 99,\ 111\times 99,\ 202\times 99,\ 212\times 99,\ 222\times 99,\\ &303\times 99,\ 313\times 99,\ 323\times 99,\ 333\times 99,\\ &\vdots\\ &909\times 99,\ 919\times 99,\ 929\times 99,\ldots,999\times 99. \end{aligned}$$

Moreover, each T_5-orbit must be the subset of the above set of numbers.

Remark 1. *We have $T_5(99\times 111) = T_5(99 \times 999)$.*

Theorem 4. *Each orbit of operator T_7 must contain only the numbers of the form*

$$AB(A+C)BA \times 99, \tag{2}$$

where $0 \leq C \leq B \leq A \leq (A+C) \leq 9$, or

$$A(B+1)(A+C-10)BA \times 99, \tag{3}$$

where $1 \leq C \leq B \leq A \leq 9 < (A+C)$ and $B \leq 8$.

Proof: Let n be the seven-digit number composed of the following seven digits

$$0 \leq g \leq f \leq e \leq d \leq c \leq b \leq a \leq 9.$$

Then we have

$$\begin{aligned} &T_7(n) = (a-g)(10^6-1)+(b-f)(10^5-10)+\\ &+(c-e)(10^4-10^2) = 99\times((a-g)\times 10101+\\ &+(b-f)\times 1010+(c-e)\times 100) =\\ &= \begin{cases} AB(A+C)BA\times 99, & \text{if } A+C \leq 9,\\ A(B+1)(A+C-10)BA\times 99, & \text{if } A+C>9 \\ & \text{and } B\leq 8, \end{cases} \end{aligned}$$

where $A := a-g$, $B := b-f$, $C := c-e$. It is obvious that we have $0 \leq C \leq B \leq A \leq 9$. ■

Corollary 2. *Orbits of operator T_7 can be sought only from among the T_7-compositions on the following numbers (we give first the numbers defined by formula (2)):*

$$\begin{aligned} &10101\times 99,\ 11111\times 99,\ 11211\times 99,\\ &20202\times 99,\ 21212\times 99,\ 21312\times 99,\\ &22222\times 99,\ 22322\times 99,\ 22422\times 99,\\ &30303\times 99,\ldots,\\ &\vdots\\ &90909\times 99,\ldots,98989\times 99,\ 999999\times 99, \end{aligned}$$

and (it is about 163 numbers described by formula (3)):

$$\begin{aligned} &99089\times 99,\ 99189\times 99,\ldots,99889\times 99,\\ &98079\times 99,\ 98179\times 99,\ldots,98679\times 99,\\ &\vdots\\ &93029\times 99,\ 93129\times 99,\\ &92019\times 99,\\ &89088\times 99,\ 89188\times 99,\ldots,89688\times 99,\\ &88078\times 99,\ 88178\times 99,\ldots,88578\times 99,\\ &\vdots\\ &84038\times 99,\ 84138\times 99,\\ &83028\times 99,\\ &\vdots\\ &67066\times 99,\ 67166\times 99,\ 67266\times 99,\\ &66056\times 99,\ 66156\times 99,\\ &65046\times 99,\\ &56055\times 99. \end{aligned}$$

Theorem 5. *Each orbit of operator T_6 contains only the numbers described by the following seven formulae*

$$9\times A(A+B)(A+B+C)(A+B)A, \tag{4}$$

where $0 \leq C \leq B \leq A \leq A+B+C \leq 9$, or

$$9\times A(A+B+1)(A+B+C-10)(A+B)A, \tag{5}$$

where $0 \leq C \leq B \leq A \leq A+B \leq 8$ and $10 \leq A+B+C < 20$, or

$$9\times (A+1)0(A+B+C-10)9A, \tag{6}$$

where $1 \leq C \leq B \leq A \leq 9$ and $A+B = 9$, or

$$9\times (A+1)(A+B-9)(A+B+C-9)(A+B-10)A, \tag{7}$$

where $0 \leq C \leq B \leq A \leq 9$ and $10 \leq A+B \leq A+B+C \leq 18$, or

$$9\times (A+1)(A+B-8)(A+B+C-19)(A+B-10)A, \tag{8}$$

where $0 \le C \le B \le A \le 9$ and $A+B+C \ge 19$ (we note that then $A+B \ge 10$) and $A+B \le 17$, or

$$9 \times 110(C-1)89, \tag{9}$$

where $C \ge 1$, or

$$9 \times 109989. \tag{10}$$

Proof: In order to get the presented formulae let us assume that n is the natural six-digits number composed of the digits $0 \le a_6 \le a_5 \ldots \le a_1 \le 9$. Then we obtain

$$T_6(n) = 9((a_1-a_6)(10^4+10^3+10^2+10+1)+ \\ +(a_2-a_5)(10^3+10^2+10)+(a_3-a_4)10^2).$$

By taking $A := a_1-a_6$, $B := a_2-a_5$, $C := a_3-a_4$ we find

$$T_6(n) = A10^4+(A+B)10^3+(A+B+C)10^2+(A+B)10+A,$$

where $0 \le C \le B \le A \le 9$. The only thing which left is to analyze the value of sums $A+B+C$ and $A+B$ which gives the thesis of theorem. ■

Remark 2. *Although we have as many as seven different formulae describing potential numbers belonging to the orbits of operator T_6, their description can be directly generated in easy way. However, we will omit here this description.*

Remark 3. *It was numerically proved by the Authors that operator T_6 possesses three fixed points (the Kaprekar's constants of sixth order):*

$$0,\ 549945,\ 631764$$

and one 7-element orbit (we give it in a table in Part II of this paper). The information on an existing 7-element orbit is omitted in the table presented in the Polish version of Wikipedia (http://pl.wikipedia.org/wiki/Stala_Kaprekara).

Theorem 6. *Orbits of operator T_4 contain only the numbers described by formulae*

$$9 \times A(A+B)A, \tag{11}$$

where $0 \le B \le A \le A+B \le 9$, or

$$9 \times (A+1)(A+B-10)A, \tag{12}$$

where $1 \le B \le A \le 9$ and $A+B \ge 10$.

Proof: Let $n \in \mathbb{N}$ be the four-digit number composed of digits $0 \le d \le c \le b \le a \le 9$. Then we have

$$\begin{aligned} T_4(n) &= (a-d)(10^3-1)+(b-c)(10^2-10) = \\ &= 9 \times \big((a-d)(10^2+10+1)+(b-c)10\big) = \\ &= \begin{cases} 9 \times A(A+B)A, & \text{if } A+B \le 9, \\ 9 \times (A+1)(A+B-10)A, & \text{if } A+B > 9, \end{cases} \end{aligned}$$

where $A := a-d$, $B := b-c$. Certainly we have $0 \le B \le A \le 9$. ■

Remark 4. *Formulae (11) and (12) describe the following 45 numbers*

$$\begin{aligned} &111 \times 9,\ 121 \times 9, \\ &222 \times 9,\ 232 \times 9,\ 242 \times 9, \\ &333 \times 9,\ 343 \times 9,\ 353 \times 9,\ 363 \times 9, \\ &444 \times 9,\ 454 \times 9, \ldots, 484 \times 9, \\ &555 \times 9,\ 565 \times 9, \ldots, 595 \times 9, \\ &605 \times 9, \\ &666 \times 9,\ 676 \times 9,\ 686 \times 9^*,\ 696 \times 9, \\ &706 \times 9,\ 716 \times 9,\ 726 \times 9, \\ &777 \times 9,\ 787 \times 9,\ 797 \times 9, \\ &807 \times 9,\ 817 \times 9, \ldots, 847 \times 9, \\ &888 \times 9,\ 898 \times 9,\ 908 \times 9, \ldots, 968 \times 9,\ 999 \times 9. \end{aligned}$$

where by * we have distinguished the Kaprekar's constant. Directly calculating (even by hand – if we are extremely dogged) we can verify that T_4 possesses only one orbit

$$\{686 \times 9 = 6174\}.$$

Let us recall, that this fixed point of T_4, i.e. number 6174, is called the Kaprekar's constant (of fourth order).

III. Final remarks

Authors of this paper, apart from the discussed here Kaprekar's transformations, have also defined and investigated the minimal orbits (cycles, respectively) of few generalizations of these transformations, like for example

— the symmetric Kaprekar's transformation
Let $a_1a_2\ldots a_n$ be the decimal expansion of number $a \in \mathbb{N}$, $10^{n-1} \le a < 10^n$. Then the n-th symmetric Kaprekar's transformation M is defined as

$$M(a_1a_2\ldots a_n) = \sum_{k=1}^{n} |c_k - b_k| 10^{k-1}$$

where $(b_1, b_2, \ldots, b_n)$ and $(c_1, c_2, \ldots, c_n)$ are the sequences, nondecreasing and nonincreasing, respectively, composed of the digits $a_1, a_2, ..., a_n$. We include to the set of n-digit numbers also the number zero. Orbits of operators M for the odd values $n \le 19$, although "quite easy" to calculate even by hand, surprise yet with their final form. We will present here only few quantitative pieces of information.
So, if $n = 2k+1$, $1 \le k \le 5$, then M possesses only the fixed points and k-element orbits, for $n = 13$ operator M possesses two fixed points, 0 and 65432101…6, four 2-element cycles, eleven 3-element cycles and 827 cycles of length 6 (sic). For $n = 15$ the operator M possesses 44 fixed points, 342 different 2-elements orbits and 2678 different 4-elements orbits. For $n = 17$ the operator M possesses only 6 fixed points, 32 different 2-element orbits and 6060 different 4-element orbits. Finally, for $n = 2^k$ the operator M possesses only trivial orbit $= \{0\}$ for every $k \in \mathbb{N}$.

— nonoptimal Kaprekar's transformations
One of the examples of this transformation, called by us the Q-Kaprekar's transformation, is defined as

$$Q_n(A) := (a_n - a_2)10^{n-1} + (a_{n-1} - a_1)10^{n-2} + \\ + \sum_{k=1}^{n-2}(a_k - a_{n-k+1})10^{k-1},$$

where $0 \leq a_1 \leq a_2 \leq \ldots \leq a_n \leq 9$ are the all digits of decimal expansion of number A. We note that, in contrast to the Kaprekar's transformation T_4, the transformation Q_4 possesses two 2-element orbits: $\{2187, 6543\}$ and $\{3285, 5274\}$ and the trivial fixed point. Next, Q_5 possesses the trivial fixed point and the 2-element orbit $\{52974, 54963\}$ (in contrast, transformation T_5 has four different orbits). Transformations Q_6 and T_6 have both three fixed points and, respectively, the 8-element orbit and the 7-element orbit. Transformations Q_7 and T_7 possess both the trivial fixed point and one 8-element orbit (but of different orbit types).

— general Kaprekar's transformations
We take that the natural number A, $10^{n-1} \leq A < 10^n$, possesses the following decimal expansion $A = d_1 d_2 \ldots d_n$. Let $a_1 := \max\{d_1, d_2, \ldots, d_n\}$, $a_2 := \max\{d_2, d_3, \ldots, d_n\}$ and in general $a_k := \max\{d_k, d_{k+1}, \ldots, d_n\}$, for $k = 1, 2, \ldots, n$. The announced general Kaprekar's transformations are defined by relations

$$d_{\sigma,\pi}(A) := \sum_{k=1}^{n} |d_{\sigma(k)} - d_{\pi(k)}| 10^{n-k},$$

$$d_{\sigma,\pi}^{weak}(A) := \left| \sum_{k=1}^{n} (d_{\sigma(k)} - d_{\pi(k)}) 10^{n-k} \right|,$$

and

$$D_{f,g}(A) := \sum_{k=1}^{n} |d_{f(k)} - d_{g(k)}| 10^{n-k},$$

$$D_{f,g}^{weak}(A) := \left| \sum_{k=1}^{n} (d_{f(k)} - d_{g(k)}) 10^{n-k} \right|,$$

$$R_f(A) := \sum_{k=1}^{n} |a_k - a_{f(k)}| 10^{n-k},$$

$$R_f^{weak}(A) := \left| \sum_{k=1}^{n} (a_k - a_{f(k)}) 10^{n-k} \right|,$$

for any permutations σ, π on set $\{1, 2, \ldots, n\}$ and for any functions $f, g\colon \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$.

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