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Kaprekar's transformations. Part II – numerical results and intriguing corollaries

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***Abstract*—This paper is a continuation of our previous paper [Part I, ibidem]. In this study we present many new results in the subject of minimal cycles (including the fixed points) of the so called Kaprekar's transformations. We formulate also some conjectures. Moreover, we discuss here all minimal cycles of the first 18 Kaprekar's transformations (and present but only of the first 15) with emphasis of the new, introduced by us, characteristics of this cycles.**

I. Introduction

In Part I of this elaboration (see [1]) we have introduced the definitions of the so called Kaprekar's transformations T_n:

$$\begin{aligned} T_n\colon \{0\}\cup\big\{\alpha\colon 10^{n-1}-1\leq\alpha<10^n\big\} &\to \\ \to \{0\}\cup\big\{\alpha\colon 10^{n-1}-1\leq\alpha<10^n\big\} & \\ T_n(\alpha):=\sum_{k=1}^{n}(a_k-a_{n-k+1})10^{k-1} & \\ = a_n a_{n-1}\ldots a_1 - a_1 a_2\ldots a_n, & \end{aligned}$$

for every $\alpha, n\in\mathbb{N}$, $10^{n-1}-1\leq\alpha<10^n$, where

$$0\leq a_1\leq a_2\leq\ldots\leq a_n\leq 9,$$

denote all digits of decimal expansion of number α ordered in nondecreasing sequence and $T_n(0)=0$. We have also described the orbits of maps T_n for $n=3,4,\ldots,7$. Furthermore, in Part I many new concepts and characteristics of the minimal cycles of general transformations $F\colon X\to X$, where X is a finite set, have been proposed. All of them will be used in this part of our paper and applied for the Kaprekar's transformations T_n, $n\in\mathbb{N}$.

Moreover, in this part of our paper we intend to present firstly the collection of absolutely new facts discovered by observing the, numerically obtained, orbits of operators T_n for $n\leq 18$. Next we will compile in tables the detailed descriptions of the minimal cycles of operators T_n for $n\leq 15$ (that is, we will give many individual pieces of information concerning each of the investigated cycles). The other cases for $n=16-18$, because of the permissible length of the paper, are omitted here.

II. Facts basing on the numerical results

Let us present now several essential facts in the subject of Kaprekar's transformations which we have deduced by analyzing the numerically obtained minimal cycles of operators T_n for $n\leq 18$. We will also formulate some conjectures concerning the cycles of Kaprekar's transformations.

Fact 1. *Numbers appearing in the orbits of transformations T_n correspond with the partitions of number $\lceil\frac{n}{2}\rceil\times 9$ into n digits, except the following $n=3k$-digit numbers being the Kaprekar's constants of order $3k$ with the sum of digits equal to $18k$:*

$$495,\ 549945,\ 554999445,\ \ldots,\ \underbrace{5\ldots5}_{(k-1)\ digits}\ 4\underbrace{9\ldots9}_{k\ digits}\ \underbrace{4\ldots4}_{(k-1)\ digits}\ 5.$$

The following theorem and the respective conclusions constitute the theoretical grounds of the described above properties of the orbits of transformations T_n.

Theorem 1.

a) *Let $a\in\mathbb{N}$ be a $2n$-digit number composed of digits*
$$0\leq a_1\leq a_2\leq\ldots\leq a_{2n}\leq 9$$
and suppose that
$$a_{n-k-1}<a_{n-k}=a_{n-k+1}=\ldots=a_{n+l}<a_{n+l+1}$$
for some $k,l\in\mathbb{N}_0$.
If $k\geq l$, then the sum of digits of number $T_{2n}(a)$ is equal to $9\times(n+l)$. Otherwise, this sum is equal to $9\times(n+k)$.

b) *Let $a\in\mathbb{N}$ be a $(2n+1)$-digit number composed of digits*
$$0\leq a_1\leq a_2\leq\ldots\leq a_{2n+1}\leq 9$$
and suppose that
$$a_{n-k}<a_{n-k+1}=a_{n-k+2}=\ldots=a_{n+l+1}<a_{n+l+2}$$
for some $k,l\in\mathbb{N}_0$.
If $k\geq l$, then the sum of digits of number $T_{2n+1}(a)$ is equal to $9\times(n+l+1)$, whereas if $k<l$, then the sum of digits of number $T_{2n+1}(a)$ is equal to $9\times(n+k+1)$.

Proof:
ad a) Let us notice that the following decimal expansions of $T_{2n}(a)$ can be obtained

$$T_{2n}(a)=\begin{cases}(a_{2n}-a_1)(a_{2n-1}-a_2)\ldots(a_{n+k+1}-a_{n-k}-1)\\ \times(9+a_{n+k}-a_{n-k})\ldots(9+a_2-a_{2n-1})\\ \times(10+a_1-a_{2n}),\ \text{if } l>k,\\ (a_{2n}-a_1)(a_{2n-1}-a_2)\ldots(a_{n+l+1}-a_{n-l}-1)\\ \times(9+a_{n+l}-a_{n-l})\ldots(9+a_2-a_{2n-1})\\ \times(10+a_1-a_{2n}),\ \text{if } k\geq l,\end{cases}$$

which implies the assertion.
ad b) The proof runs in similar way as in case of item a). ■

Corollary 1. *If $a \in \mathbb{N}$ is a n-digit number then the sum of digits of number $T_n(a)$ is not lower than the number $9 \times \lceil \frac{n}{2} \rceil$.*

Corollary 2. *If $a \in \mathbb{N}$ is a number possessing different digits in the decimal expansion then the sum of digits of number $T_n(a)$ is equal to $9 \times \lceil \frac{n}{2} \rceil$.*

Conjecture 1. *Sum of digits of the numbers belonging to the given orbit of operator T_n, where $n \in \mathbb{N}$, except the two-element orbit of operator T_5, is the same.*

Remark 1. *The lowest number n, for which there exist two different orbits (two different orbits possessing at least two elements) of operator T_n composed of the numbers with different sums of digits, is equal to 6 (is equal to $n = 16$, respectively).*

Remark 2. *Numbers belonging to the orbits of operator T_{2n+1} possess in their decimal expansion the middle digit equal to 9.*

Fact 2. *Let $a_1 a_2 \ldots a_n$ be an n-digit number belonging to some orbit of transformation T_n, $n \in \mathbb{N}$. Then the sequence, henceforward called as the digit type of element $a_1 a_2 \ldots a_n$ of the given cycle, defined in the following way*

$$a_1+a_n, a_2+a_{n-1}, a_3+a_{n-2}, \ldots, \begin{cases} a_{\frac{n}{2}} + a_{\frac{n}{2}+1}, & \text{if } n \text{ is even} \\ a_{\frac{n+1}{2}}, & \text{if } n \text{ is odd} \end{cases}$$

is equal to

$$10, \underbrace{9, \ldots, 9,}_{(k-1)\text{-times}} 8, 9 \tag{1}$$

if $n = 2k+1$, $k = 1, 2, \ldots$, and

$$10, \underbrace{9, \ldots, 9,}_{(k-2)\text{-times}} 8 \tag{2}$$

if $n = 2k$, $k = 2, 3, \ldots$ In both cases the equality holds independently on number $a_1 a_2 \ldots a_n$, except the following numbers:

(i) *the Kaprekar's constants of order $n = 3k$:*

$$\underbrace{5 \ldots 5}_{(k-1)\text{-times}} \ 4 \underbrace{9 \ldots 9}_{k\text{-times}} \ \underbrace{4 \ldots 4}_{(k-1)\text{-times}} \ 5,$$

for which the respective sequence of sums has the form

$$10, \underbrace{9, \ldots, 9,}_{(k-2)\text{-times}} 8, \underbrace{18, \ldots, 18,}_{\lfloor \frac{k}{2} \rfloor\text{-times}} \underbrace{9.}_{(\lceil \frac{k}{2} \rceil - \lfloor \frac{k}{2} \rfloor)\text{-times}}$$

Let us notice that if we correct the above sequence in the following way (we shift the units similarly as in the addition operation):

$$10, \underbrace{9, \ldots, \ \ 9,}_{(k-2)\text{-times}} 8, \underbrace{18, 18, \ldots, \ 18,}_{\lfloor \frac{k}{2} \rfloor\text{-times}} \underbrace{9.}_{(\lceil \frac{k}{2} \rceil - \lfloor \frac{k}{2} \rfloor)\text{-times}}$$

then we obtain the sequence

$$10, \ \underbrace{9, \ldots, 9}_{(\lfloor \frac{3k}{2} \rfloor - 2)\text{-times}} \ 8, \ \underbrace{9,}_{(\lceil \frac{k}{2} \rceil - \lfloor \frac{k}{2} \rfloor)\text{-times}}$$

which is "compatible" either with (1), if k is odd, or with (2), if k is even.

(ii) *the numbers belonging to the single 2-element orbit $\{53955, 59994\}$ of operator T_5, where the respective sequences of sums are of the forms $10, 8, 9$ and $9, 18, 9$, but we get*

$$9, 18, 9 \longmapsto 10, 8, 9.$$

(iii) *the numbers belonging to the single 2-element orbit $\{8764421997755322, 8765431997654322\}$ of operator T_{16}, where both sequences of sums are of the form $10, 9, 9, 9, 9, 9, 8, 18$, but we obtain*

$$10, 9, 9, 9, 9, 9, 8, 18 \longmapsto 10, 9, 9, 9, 9, 9, 9, 8,$$

which is compatible with (2).

Fact 3. *We have noticed that for every $n = 10, 12, \ldots, 18$ the operator T_n possesses the even number of 3-element cycles and, moreover, the difference between the numbers of 3-element cycles of T_n possessing the orbit types $(1,3,2)$ and $(1,2,3)$, respectively, is equal to 0 for $n = 10, 12$ and $2^{\frac{n-12}{2}}$ for $n = 14, 16, 18$. The orbit type of all 7-element cycles of T_n, $n \leq 18$, is the same and is equal to $(1,5,3,4,6,7,2)$.*

Fact 4 (Kaprekar's constants)**.** *We have observed that each Kaprekar's constant of order $n \leq 18$ generates the sequence of extensions of decimal expansions remaining the Kaprekar's constants (of the respectively higher order). For example, we have*

— $6\underbrace{3 \ldots 3}_{k\text{-times}}17\underbrace{6 \ldots 6}_{k\text{-times}}4$ *are the Kaprekar's constants of order $(2k+4)$ for every $k = 0, 1, 2, \ldots$,*

Sketch of the proof: We have

$$7\underbrace{6 \ldots 6}_{k+1}4\underbrace{3 \ldots 3}_{k}1 - 1\underbrace{3 \ldots 3}_{k}4\underbrace{6 \ldots 6}_{k+1}7 = 6\underbrace{3 \ldots 3}_{k}17\underbrace{6 \ldots 6}_{k}4$$

■

— $\underbrace{9 \ldots 9}_{k\text{-times}}750842\underbrace{0 \ldots 0}_{(k-1)\text{-times}}1$ *are the Kaprekar's constants of order $(2k+6)$ for every $k = 1, 2, \ldots$,*

— $975\underbrace{3 \ldots 3}_{k\text{-times}}08\underbrace{6 \ldots 6}_{k\text{-times}}421$ *are the Kaprekar's constants of order $(2k+8)$ for every $k = 0, 1, 2, \ldots$,*

— $\underbrace{9 \ldots 9}_{k\text{-times}}75308642\underbrace{0 \ldots 0}_{(k-1)\text{-times}}1$ *are the Kaprekar's constants of order $(2k+8)$ for every $k = 1, 2, \ldots$,*

— $864\underbrace{3 \ldots 3}_{k\text{-times}}197\underbrace{6 \ldots 6}_{k\text{-times}}532$ *are the Kaprekar's constants of order $(2k+9)$ for every $k = 0, 1, 2, \ldots$*

Remark 3. *The Q-Kaprekar's transformations Q_n, defined in the last section of Part I, possess the same property as above for their fixed points. For example, the number*

$$\underbrace{5\dots5}_{k\text{-times}}4\underbrace{9\dots9}_{(k+1)\text{-times}}\underbrace{4\dots4}_{k\text{-times}}5$$

is the fixed point of transformation Q_{3k+3} for every $k = 1, 2, \dots$, the number

$$66\underbrace{3\dots3}_{k\text{-times}}08\underbrace{6\dots6}_{k\text{-times}}52$$

is the fixed point of Q_{2k+6} for every $k = 0, 1, 2, \dots$ and, at last, the number

$$\underbrace{9\dots9}_{(k+1)\text{-times}}750842\underbrace{0\dots0}_{k\text{-times}}1$$

is the fixed point of Q_{2k+8} for every $k = 1, 2, \dots$.

Fact 5. *We suppose that, similarly like in case of the Kaprekar's constants, all orbits of operators T_n with the odd number of elements possess their "extensions", that is they generate the infinite sequences of orbits of the Kaprekar's operators preserving the number of elements of the initial orbit. Whereas, despite of the insistent efforts we did not manage to get such extension (in the similar style as in case of the orbits presented below) for any orbit having the even number of elements.*

The Kaprekar's transformation $T_{2(k+4)}$, for $k = 0, 1, \dots, 5$, possesses $A140226(k)$ (equal to $\frac{1}{3}k(11 + k^2)$ for $k \geq 1$) of 3-element minimal cycles ($A140226$ in notation of the Sloane's OEIS).

Furthermore, transformation $T_{2(k+4)}$, for each $k = 0, 1, \dots,$ possesses the following 3-element minimal cycle

$$\Big(643\underbrace{3\dots3}_{k\text{-times}}08\underbrace{6\dots6}_{k\text{-times}}654,\quad 83\underbrace{3\dots3}_{k\text{-times}}2087\underbrace{6\dots6}_{k\text{-times}}62,\quad 865\underbrace{3\dots3}_{k\text{-times}}26\underbrace{6\dots6}_{k\text{-times}}432\Big).$$

For $k = 0$ it is the single 3-element minimal cycle of the respective Kaprekar's transformation.

The other examples of 3-element minimal cycles of maps T_{6k+8}, T_{2k+10}, T_{2k+10}, are the following:

$$\Big(8\underbrace{7\dots7}_{k\text{-times}}\underbrace{3\dots3}_{2k\text{-times}}32087\underbrace{6\dots6}_{2k\text{-times}}62\underbrace{2\dots2}_{k\text{-times}},\quad 865\underbrace{5\dots5}_{k\text{-times}}\underbrace{3\dots3}_{2k\text{-times}}26\underbrace{6\dots6}_{2k\text{-times}}\underbrace{4\dots4}_{k\text{-times}}432,\quad 643\underbrace{3\dots3}_{2k\text{-times}}\underbrace{1\dots1}_{k\text{-times}}08\underbrace{8\dots8}_{k\text{-times}}\underbrace{6\dots6}_{2k\text{-times}}654\Big),$$

$$\Big(975\underbrace{3\dots3}_{k\text{-times}}1088\underbrace{6\dots6}_{k\text{-times}}421,\quad 9775\underbrace{3\dots3}_{k\text{-times}}08\underbrace{6\dots6}_{k\text{-times}}4221,\quad 9755\underbrace{3\dots3}_{k\text{-times}}08\underbrace{6\dots6}_{k\text{-times}}4421\Big),$$

$$\Big(975\underbrace{5\dots5}_{k\text{-times}}1088\underbrace{4\dots4}_{k\text{-times}}421,\quad 9775\underbrace{1\dots1}_{k\text{-times}}08\underbrace{8\dots8}_{k\text{-times}}4221,\quad 97\underbrace{7\dots7}_{k\text{-times}}550844\underbrace{2\dots2}_{k\text{-times}}21\Big),$$

respectively, for every $k = 0, 1, 2, \dots$

Every Kaprekar's transformation T_{2k+11}, for $k = 0, 1, 2, \dots$, possesses the following 5-element minimal cycle

$$(864\ \underbrace{3\dots3}_{k-times}\ 20987\ \underbrace{6\dots6}_{k-times}\ 532,\quad 9664\ \underbrace{3\dots3}_{k-times}\ 197\ \underbrace{6\dots6}_{k-times}\ 5331,\quad 8843\ \underbrace{3\dots3}_{k-times}\ 197\ \underbrace{6\dots6}_{k-times}\ 6512,\quad 8764\ \underbrace{3\dots3}_{k-times}\ 197\ \underbrace{6\dots6}_{k-times}\ 5322,\quad 8654\ \underbrace{3\dots3}_{k-times}\ 197\ \underbrace{6\dots6}_{k-times}\ 5432).$$

For $k = 0$ it is the single 5-element minimal cycle of the respective Kaprekar's transformation.

Next, the transformation T_{2k+13}, for every $k = 0, 1, 2, \dots$, has also two following 5-element minimal cycles (all these cycles possess the same orbit type equal to $(1, 4, 5, 3, 2)$ and $(1, 4, 2, 5, 3)$, respectively):

$$(8654\ \underbrace{3\dots3}_{k-times}\ 20987\ \underbrace{6\dots6}_{k-times}\ 5432,\quad 9664\ \underbrace{3\dots3}_{k-times}\ 20987\ \underbrace{6\dots6}_{k-times}\ 5331,\quad 98643\ \underbrace{3\dots3}_{k-times}\ 197\ \underbrace{6\dots6}_{k-times}\ 65311,\quad 88743\ \underbrace{3\dots3}_{k-times}\ 197\ \underbrace{6\dots6}_{k-times}\ 65212,\quad 87654\ \underbrace{3\dots3}_{k-times}\ 197\ \underbrace{6\dots6}_{k-times}\ 54322)$$

and

$$(8764\ \underbrace{3\dots3}_{k-times}\ 20987\ \underbrace{6\dots6}_{k-times}\ 5322,\quad 96654\ \underbrace{3\dots3}_{k-times}\ 197\ \underbrace{6\dots6}_{k-times}\ 54331,\quad 8843\ \underbrace{3\dots3}_{k-times}\ 20987\ \underbrace{6\dots6}_{k-times}\ 6512,\quad 97664\ \underbrace{3\dots3}_{k-times}\ 197\ \underbrace{6\dots6}_{k-times}\ 53321,\quad 88543\ \underbrace{3\dots3}_{k-times}\ 197\ \underbrace{6\dots6}_{k-times}\ 65412).$$

For $k = 0$ three above 5-element minimal cycles are the only 5-element minimal cycles of T_{2k+13}.

The example of 7-element cycle of map T_{2k+6}, for every $k = 0, 1, 2, \dots$, is the following (which possesses the orbit type *equal to* $(1,5,3,4,6,7,2)$*):*

$$\Big(4\underbrace{3\ldots3}_{k\text{-times}}2087\underbrace{6\ldots6}_{k\text{-times}}6,\quad 85\underbrace{3\ldots3}_{k\text{-times}}17\underbrace{6\ldots6}_{k\text{-times}}42,$$
$$75\underbrace{3\ldots3}_{k\text{-times}}08\underbrace{6\ldots6}_{k\text{-times}}43,\quad 84\underbrace{3\ldots3}_{k\text{-times}}08\underbrace{6\ldots6}_{k\text{-times}}52,$$
$$86\underbrace{3\ldots3}_{k\text{-times}}08\underbrace{6\ldots6}_{k\text{-times}}32,\quad 86\underbrace{3\ldots3}_{k\text{-times}}26\underbrace{6\ldots6}_{k\text{-times}}32,$$
$$64\underbrace{3\ldots3}_{k\text{-times}}26\underbrace{6\ldots6}_{k\text{-times}}54\Big).$$

Indicated number 4, at the end of the last number in this cycle, appears only for $k \geq 1$.

For each $k \leq 6$ *this is the single 7-element minimal cycle of these Kaprekar's transformations.*

Fact 6. *The following statements hold for every* $n \leq 20$.

If T_n *possesses a cycle with the odd number of elements, then it possesses also a fixed point.*

Moreover, we note that there exists $n \leq 20$ *such that the operator* T_n *possesses only the nontrivial orbits with the even numbers of elements, for example we may consider* T_5*,* T_7*.*

Fact 7. *If* a *is an element belonging to the orbit of operator* T_n *composed of at least three numbers and* $a = \alpha_1\alpha_2\ldots\alpha_n$ *and* $T_n(a) = \beta_1\beta_2\ldots\beta_n$ *are the decimal representations of numbers* a *and* $T_n(a)$*, respectively, then* $\alpha_k - \beta_k = \beta_{n-k+1} - \alpha_{n-k+1}$ *for every* $k = 1,2\ldots,n$*. For example, for the cycles of operator* T_5 *(only two 4-element cycles are taken into account) we consider the following sequences of differences*

$$\beta_1-\alpha_1,\ \beta_2-\alpha_2,\ \ldots,\ \beta_5-\alpha_5.$$

Thus, for the cycle

$$(62964 = a = T_5^4(a),\quad 71973 = T_5(a),$$
$$83952 = T_5^2(a),\quad 74943 = T_5^3(a))$$

we have

$$\underbrace{-1,-2,0,2,1}_{T_5^4(a)-T_5^3(a)};\quad \underbrace{1,-1,0,1,-1}_{T_5(a)-a};$$
$$\underbrace{1,2,0,-2,-1}_{T_5^2(a)-T_5(a)};\quad \underbrace{-1,1,0,-1,1}_{T_5^3(a)-T_5^2(a)},$$

whereas for the cycle

$$(61974, 82962, 75933, 63954)$$

we have

$$0,-2,0,2,0;\ 2,1,0,-1,-2;\ -1,3,0,-3,1;\ -1,-2,0,2,1.$$

III. Conclusions

Although one can find quite a lot of references concerning the subject of the discussed here Kaprekar's transformations (see the References in [1]), we have noticed yet several lacks in descriptions of the orbits of T_n transformations, even for $n \leq 10$. Aim of our work was to complete these lacks, in which we succeeded, and we did even more. Our achievements have been indicated and included in Section II. One should emphasize especially the theorems concerning the possibility of "expanding" the fixed points and cycles of a given Kaprekar's transformation T_n, $n \leq 18$, to the fixed points and cycles of infinitely many Kaprekar's transformations (which, by the way, gives the answer to a question whether there exist infinitely many $n \in \mathbb{N}$ such that T_n possesses a fixed point - similar fact concerns the possession of 3,5,7-element orbits). For our research we introduced several new concepts which, in the context of obtained numerical results, brought us to some theoretical results and conjectures. We derived some of our theorems and conjectures presented in Section II also for the generalizations of Kaprekar's transformations (obeying the Q-Kaprekar's transformation from [1]) which will be the subject of the created now next paper. We intend also to use the experience, gained by applying the numerical results in theory, in didactic work by showing to the students the possibilities of seemingly simple calculations. We will also use in this field the experiences of other authors (see [2], [3]).

Appendix

Description of tables presenting the cycles of Kaprekar's transformations T_n

The table is composed in the following way

- in the first row the value of index n of the Kaprekar's transformation T_n is given,
- the second row presents the amount of minimal cycles of the given length of the given transformation T_n as well as the information whether the given transformation preserves the strong Sharkovsky's order or the Sharkovsky's order (see definitions 1 and 2 in [1]),
- the third row shows how many n-digit numbers is transformed by the given Kaprekar's transformation T_n (after the finite number of steps) onto the respective minimal cycle of this transformation,
- in the successive rows the successive cycles from the third row (except the trivial one, that is the zero cycle) are associated with: the order types (it concerns only the cycles of length greater than 1, see the proper definition in [1]); the sum of digits of particular elements of the cycle, in case when these sums are identical, we include them only once; the digit types, and again, in case when they are identical, we include them only once; the longest increasing interval of the given cycle, the longest increasing subsequence of the given cycle, the longest decreasing interval of the given cycle and the longest decreasing subsequence of the given cycle.

References

[1] E. Hetmaniok, M. Pleszczyński, I. Sobstyl, R. Wituła, *Kaprekar's transformations I – theoretical discussion*, ibidem.

[2] U. Świerczynska-Kaczor and J. Wachowicz, "Student Response to Educational Games - An Empirical Study", *Proc. FedCSIS* (19th Conference on Knowledge Acquisition and Management), 2013, pp.1293-1299.

[3] N. S. Papaspyrou, S. Zachos, "Teaching programming through problem solving: The role of the programming language", *Proc. FedCSIS*, 2013, pp.1533-1536.

$n=5$				
	1 fixed point, 1 cycle of length 2, 2 cycles of length 4; strong Sharkovsky's order			
	3190 numbers $\to$ cycle: (53955,59994) 48480 numbers $\to$ cycle: (61974,82962,75933,63954) 48320 numbers $\to$ cycle: (62964,71973,83952,74943)			
successive cycles	order type	sum of digits	digit type	longest incr. interval, subseq., longest decr. interval, subseq.
β_1	$(1,2)$	$27,36$	$(10,8,9)$, $(9,18,9)$	2, 1, 2, 1
β_2	$(1,4,3,2)$	27	$(10,8,9)$	2, 2, 3, 3
β_3	$(1,2,4,3)$	27	$(10,8,9)$	3, 3, 2, 2

$n=6$				
	3 fixed points, 1 cycle of length 7; Sharkovsky's order			
	1950 numbers $\to$ fixed point: 549945 62520 numbers $\to$ fixed point: 631764 935520 numbers $\to$ cycle: (420876,851742,750843,840852,860832,862632,64265)			
successive cycles	order type	sum of digits	digit type	longest incr. interval, subseq., longest decr. interval, subseq.
β_1		36	$(10,8,18)$	
β_2		27	$(10,9,8)$	
β_3	$(1,5,3,4,6,7,2)$	27	$(10,9,8)$	4, 5, 2, 3

$n=7$				
	1 fixed point, 1 cycle of length 8			
	9999990 numbers $\to$ cycle: (7509843,9529641,8719722,8649432,7519743,8429652,7619733,8439552)			
successive cycles	order type	sum of digits	digit type	longest incr. interval, subseq., longest decr. interval, subseq.
β_1	$(1,5,7,6,8,4,3,2)$	36	$(10,9,8,9)$	2, 4, 4, 5

$n=8$				
	3 fixed points, 1 cycle of length 3, 1 cycle of length 7			
	599536 numbers $\to$ fixed point: 63317664 2371040 numbers $\to$ fixed point: 97508421 48247316 numbers $\to$ cycle: (64308654,83208762,86526432) 48782098 numbers $\to$ cycle: (43208766,85317642,75308643,84308652,86308632,86326632,64326654)			
successive cycles	order type	sum of digits	digit type	longest incr. interval, subseq., longest decr. interval, subseq.
β_1,β_2		36	$(10,9,9,8)$	
β_3	$(1,2,3)$	36	$(10,9,9,8)$	3, 3, 1, 1
β_4	$(1,5,3,4,6,7,2)$	36	$(10,9,9,8)$	4, 5, 2 ,3

$n=9$				
	3 fixed points, 1 cycle of length 14; Sharkovsky's order			
	34440 numbers $\to$ fixed point: 554999445 51389136 numbers $\to$ fixed point: 864197532 948576414 numbers $\to$ cycle: $(753098643, 954197541, 883098612, 976494321, 874197522, 865296432, 763197633, 844296552, 762098733, 964395531, 863098632, 965296431, 873197622, 865395432)$			
successive cycles	order type	sum of digits	digit type	longest incr. interval, subseq., longest decr. interval, subseq.
β_1		54	$(10,9,8,18,9)$	
β_2		45	$(10,9,9,8,9)$	
β_3	$(1,11,10,14,9,6,3,4,2,12,5,13,8,7)$	45	$(10,9,9,8,9)$	2,5,4,6

$n = 10$				
	4 fixed points, 4 cycles of length 3, 1 cycle of length 7			
	4306680 numbers $\rightarrow$ fixed point: 6333176664 644450820 numbers $\rightarrow$ fixed point: 9753086421 41045760 numbers $\rightarrow$ fixed point: 9975084201 1291432626 numbers $\rightarrow$ cycle: $(6431088654, 8732087622, 8655264432)$ 3925269288 numbers $\rightarrow$ cycle: $(6433086654, 8332087662, 8653266432)$ 1058345520 numbers $\rightarrow$ cycle: $(6543086544, 8321088762, 8765264322)$ 558293820 numbers $\rightarrow$ cycle: $(9751088421, 9775084221, 9755084421)$ 2476855476 numbers $\rightarrow$ cycle: $(4332087666, 8533176642, 7533086643, 8433086652, 8633086632, 8633266632, 6433266654)$			
successive cycles	order type	sum of digits	digit type	longest incr. interval, subseq., longest decr. interval, subseq.
$\beta_1 - \beta_3$		45	$(10, 9, 9, 9, 8)$	
β_4	$(1, 3, 2)$	45	$(10, 9, 9, 9, 8)$	2, 2, 2, 2
β_5, β_6	$(1, 2, 3)$	45	$(10, 9, 9, 9, 8)$	3, 3, 1, 1
β_7	$(1, 3, 2)$	45	$(10, 9, 9, 9, 8)$	2, 2, 2, 2
β_8	$(1, 5, 3, 4, 6, 7, 2)$	45	$(10, 9, 9, 9, 8)$	4, 5, 2 ,3
$n = 11$				
	2 fixed points, 1 cycle of length 5, 1 cycle of length 8			
	7444117296 numbers $\rightarrow$ fixed point: 86431976532 61796170458 numbers $\rightarrow$ cycle: $(86420987532, 96641975331, 88431976512, 87641975322, 86541975432)$ 30759712236 numbers $\rightarrow$ cycle: $(76320987633, 96442965531, 87320987622, 96653954331, 86330986632, 96532966431, 87331976622, 86542965432)$			
successive cycles	order type	sum of digits	digit type	longest incr. interval, subseq., longest decr. interval, subseq.
β_1		54	$(10, 9, 9, 9, 8, 9)$	
β_2	$(1, 5, 4, 3, 2)$	54	$(10, 9, 9, 9, 8, 9)$	2, 2, 4, 4
β_3	$(1, 6, 4, 8, 2, 7, 5, 3)$	54	$(10, 9, 9, 9, 8, 9)$	2, 3, 3, 4
$n = 12$				
	6 fixed points, 10 cycles of length 3, 1 cycle of length 7			
	697950 numbers $\rightarrow$ fixed point: 555499994445 57413664 numbers $\rightarrow$ fixed point: 633331766664 28903840680 numbers $\rightarrow$ fixed point: 975330866421 6771885120 numbers $\rightarrow$ fixed point: 997530864201 556839360 numbers $\rightarrow$ fixed point: 999750842001 23752825668 numbers $\rightarrow$ cycle: $(643110888654, 877320876222, 865552644432)$ 125925387258 numbers $\rightarrow$ cycle: $(643310886654, 873320876622, 865532664432)$ 250807302642 numbers $\rightarrow$ cycle: $(643330866654, 833320876662, 865332666432)$ 37978377360 numbers $\rightarrow$ cycle: $(654310886544, 873210887622, 876552644322)$ 124802255728 numbers $\rightarrow$ cycle: $(654330866544, 833210887662, 876532664322)$ 76745507520 numbers $\rightarrow$ cycle: $(655430865444, 832110888762, 877652643222)$ 14186684160 numbers $\rightarrow$ cycle: $(975110888421, 977750842221, 975550844421)$ 91728976482 numbers $\rightarrow$ cycle: $(975310886421, 977530864221, 975530864421)$ 35851244880 numbers $\rightarrow$ cycle: $(975510884421, 977510884221, 977550844221)$ 10397350260 numbers $\rightarrow$ cycle: $(997510884201, 997750842201, 997550844201)$ 171533411258 numbers $\rightarrow$ cycle: $(433320876666, 853331766642, 753330866643, 843330866652, 863330866632, 863332666632, 643332666654)$			
successive cycles	order type	sum of digits	digit type	longest incr. interval, subseq., longest decr. interval, subseq.
β_1		72	$(10, 9, 9, 8, 18, 18)$	
$\beta_2 - \beta_5$		54	$(10, 9, 9, 9, 9, 8)$	
β_6, β_7	$(1, 3, 2)$	54	$(10, 9, 9, 9, 9, 8)$	2, 2, 2, 2
$\beta_8 - \beta_{11}$	$(1, 2, 3)$	54	$(10, 9, 9, 9, 9, 8)$	3, 3, 1, 1
β_{12}, β_{13}	$(1, 3, 2)$	54	$(10, 9, 9, 9, 9, 8)$	2, 2, 2, 2
β_{14}	$(1, 2, 3)$	54	$(10, 9, 9, 9, 9, 8)$	3, 3, 1, 1
β_{15}	$(1, 3, 2)$	54	$(10, 9, 9, 9, 9, 8)$	2, 2, 2, 2
β_{16}	$(1, 5, 3, 4, 6, 7, 2)$	54	$(10, 9, 9, 9, 9, 8)$	4, 5, 2, 3

$n = 13$				
	2 fixed points, 1 cycle of length 2, 3 cycles of length 5; Sharkovsky's order			
	127766869230 numbers $\rightarrow$ fixed point: 8643319766532 729214292326 numbers $\rightarrow$ cycle: $(8733209876622, 9665429654331)$ 5169476073242 numbers $\rightarrow$ cycle: $(8643209876532, 9664319765331, 8843319766512, 8764319765322, 8654319765432)$ 1373689940636 numbers $\rightarrow$ cycle: $(8654209875432, 9664209875331, 9864319765311, 8874319765212, 8765419754322)$ 2599852824556 numbers $\rightarrow$ cycle: $(8764209875322, 9665419754331, 8843209876512, 9766419753321, 8854319765412)$			
successive cycles	order type	sum of digits	digit type	longest incr. interval, subseq., longest decr. interval, subseq.
β_1		63	$(10, 9, 9, 9, 9, 8, 9)$	
β_2	$(1, 2)$	63	$(10, 9, 9, 9, 9, 8, 9)$	2, 2, 1, 1
β_3	$(1, 5, 4, 3, 2)$	63	$(10, 9, 9, 9, 9, 8, 9)$	2, 2, 4, 4
β_4	$(1, 4, 5, 3, 2)$	63	$(10, 9, 9, 9, 9, 8, 9)$	3, 3, 3, 3
β_5	$(1, 4, 2, 5, 3)$	63	$(10, 9, 9, 9, 9, 8, 9)$	2, 3, 2, 2

$n = 14$				
	7 fixed points, 20 cycles of length 3, 1 cycle of length 7			
	825128304 numbers $\rightarrow$ fixed p.: 63333317666664; 1640938809510 numbers $\rightarrow$ fixed p.: 97533308666421 1955480289854 numbers $\rightarrow$ fixed p.: 97755108844221; 516356961120 numbers $\rightarrow$ fixed p.: 99753308664201 126071225280 numbers $\rightarrow$ fixed p.: 99975308642001; 6034588560 numbers $\rightarrow$ fixed p.: 99997508420001 616791947798 numbers $\rightarrow$ cycle: $(64311108888654, 87773208762222, 86555526444432)$ 2245517211436 numbers $\rightarrow$ cycle: $(64331108886654, 87733208766222, 86555326644432)$ 12115951630042 numbers $\rightarrow$ cycle: $(64333108866654, 87333208766622, 86553326664432)$ 20900682225326 numbers $\rightarrow$ cycle: $(64333308666654, 83333208766662, 86533326666432)$ 1233797593392 numbers $\rightarrow$ cycle: $(65431108886544, 87732108876222, 87655526444322)$ 4978650152970 numbers $\rightarrow$ cycle: $(65433108866544, 87332108876622, 87655326644322)$ 8893048070816 numbers $\rightarrow$ cycle: $(65433308666544, 83332108876662, 87653326664322)$ 1917234715396 numbers $\rightarrow$ cycle: $(65543108865444, 87321108887622, 87765526443222)$ 4466367674132 numbers $\rightarrow$ cycle: $(65543308665444, 83321108887662, 87765326643222)$ 1355384297358 numbers $\rightarrow$ cycle: $(65554308654444, 83211108888762, 87776526432222)$ 360886383858 numbers $\rightarrow$ cycle: $(97511108888421, 97777508422221, 97555508444421)$ 2896580093862 numbers $\rightarrow$ cycle: $(97531108886421, 97775308642221, 97555308644421)$ 5677743145438 numbers $\rightarrow$ cycle: $(97533108866421, 97753308664221, 97553308664421)$ 2626503498710 numbers $\rightarrow$ cycle: $(97551108884421, 97775108842221, 97755508444221)$ 6197474439338 numbers $\rightarrow$ cycle: $(97553108864421, 97753108864221, 97755308644221)$ 1366108585842 numbers $\rightarrow$ cycle: $(97555108844421, 97751108884221, 97775508442221)$ 420203255472 numbers $\rightarrow$ cycle: $(99751108884201, 99777508422201, 99755508444201)$ 2316236914992 numbers $\rightarrow$ cycle: $(99753108864201, 99775308642201, 99755308644201)$ 829988923764 numbers $\rightarrow$ cycle: $(99755108844201, 99775108842201, 99775508442201)$ 181449067800 numbers $\rightarrow$ cycle: $(99975108842001, 99977508422001, 99975508442001)$ 14157693169620 numbers $\rightarrow$ cycle: $(43333208766666, 85333317666642, 75333308666643, 84333308666652, 86333308666632, 86333326666632, 64333326666654)$			
successive cycles	order type	sum of digits	digit type	longest incr. interval, subseq., longest decr. interval, subseq.
$\beta_1 - \beta_6$		63	$(10, 9, 9, 9, 9, 9, 8)$	
$\beta_7 - \beta_9$	$(1, 3, 2)$	63	$(10, 9, 9, 9, 9, 9, 8)$	2, 2, 2, 2
β_{10}, β_{25}	$(1, 2, 3)$	63	$(10, 9, 9, 9, 9, 9, 8)$	3, 3, 1, 1
β_{11}, β_{26}	$(1, 3, 2)$	63	$(10, 9, 9, 9, 9, 9, 8)$	2, 2, 2, 2
$\beta_{12} - \beta_{16}$	$(1, 2, 3)$	63	$(10, 9, 9, 9, 9, 9, 8)$	3, 3, 1, 1
$\beta_{17} - \beta_{20}$	$(1, 3, 2)$	63	$(10, 9, 9, 9, 9, 9, 8)$	2, 2, 2, 2
β_{21}, β_{22}	$(1, 2, 3)$	63	$(10, 9, 9, 9, 9, 9, 8)$	3, 3, 1, 1
β_{23}, β_{24}	$(1, 3, 2)$	63	$(10, 9, 9, 9, 9, 9, 8)$	2, 2, 2, 2
β_{27}	$(1, 5, 3, 4, 6, 7, 2)$	63	$(10, 9, 9, 9, 9, 9, 8)$	4, 5, 2, 3

$n = 15$				
	3 fixed points, 1 cycle of length 2, 5 cycles of length 5			
	15165150 numbers $\rightarrow$ fixed p.: 555549999944445; 3577552068090 numbers $\rightarrow$ fixed p.: 864333197666532 12790914986700 numbers $\rightarrow$ cycle: $(873332098766622, 966543296654331)$ 91463039030240 numbers $\rightarrow$ cycle: $(864332098766532, 966433197665331, 884333197666512, 876433197665322, 865433197665432)$ 234193123825336 numbers $\rightarrow$ cycle: $(865432098765432, 966432098765331, 986433197665311, 887433197665212, 876543197654322)$ 270342559594928 numbers $\rightarrow$ cycle: $(876432098765322, 966543197654331, 884332098766512, 976643197653321, 885433197665412)$ 146805971092664 numbers $\rightarrow$ cycle: $(876542098754322, 966542098754331, 986432098765311, 987643197653211, 887543197654212)$ 240826824236882 numbers $\rightarrow$ cycle: $(885432098765412, 976642098753321, 986543197654311, 887432098765212, 976654197543321)$			
successive cycles	order type	sum of digits	digit type	longest incr. interval, subseq., longest decr. interval, subseq.
β_1		90	$(10, 9, 9, 9, 8, 18, 18, 9)$	
β_2		72	$(10, 9, 9, 9, 9, 9, 8, 9)$	
β_3	$(1, 2)$	72	$(10, 9, 9, 9, 9, 9, 8, 9)$	2, 2, 1, 1
β_4	$(1, 5, 4, 3, 2)$	72	$(10, 9, 9, 9, 9, 9, 8, 9)$	2, 2, 4, 4
β_5	$(1, 4, 5, 3, 2)$	72	$(10, 9, 9, 9, 9, 9, 8, 9)$	3, 3, 3, 3
β_6	$(1, 4, 2, 5, 3)$	72	$(10, 9, 9, 9, 9, 9, 8, 9)$	2, 3, 2, 2
β_7	$(1, 3, 4, 5, 2)$	72	$(10, 9, 9, 9, 9, 9, 8, 9)$	4, 4, 2, 2
β_8	$(1, 3, 5, 2, 4)$	72	$(10, 9, 9, 9, 9, 9, 8, 9)$	3, 3, 2, 2