Position Papers of the Federated Conference on Computer Science and Information Systems pp. 19–23
DOI: 10.15439/2016F564
ACSIS, Vol. 9. ISSN 2300-5963

Computing the minimal solutions of finite fuzzy relation equations on linear carriers

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Abstract—**Fuzzy relation equations is an important tool for managing and modeling uncertain or imprecise datasets, which has useful applied to, e.g. approximate reasoning, time series forecast, decision making, fuzzy control, etc. This paper considers a general fuzzy relation equation, which has minimal solutions, if it is solvable. In this case, an algebraic characterization is introduced which provides an interesting method to compute minimal solutions in this general setting.**

I. INTRODUCTION

FUZZY relation equations were introduced by E. Sanchez in the seventies [11]. These equations have widely been studied in different papers [1], [3], [6]. For example, they have proven that the set of solutions of solvable fuzzy relation equations is a upper-preserving complete lattice in which the greatest solutions is completely determined. Nevertheless, the computation of minimal solutions is not so direct. These solutions have also been studied in several papers [2], [4], [10], [12], [17], [15], [16], [18], [13] and several algorithms have been developed, but in restrictive frameworks, restrictions that limit the flexibility of the possible applications.

Hence, first of all, it is fundamental to study general frameworks in which the minimal solutions of each solvable fuzzy relation equation exist and that each solution will be between the greatest solution and a minimal solution.

This paper considers a general setting, in which the operators may neither be commutative nor associative and they only need to be monotone and residuated inf-preserving mappings of non-empty sets on the right argument. The linearity of the carrier, together with the inf-preserving property, ensures the existence of minimal solutions whenever a solution exists.

Mainly, this paper introduces a procedure in order to obtain the minimal solutions of a solvable of the introduced general fuzzy relation equations. Moreover, we have presented a detailed algorithm to compute these important solutions, together with several illustrative examples.

II. GENERAL FUZZY RELATION EQUATIONS

Throughout this paper we will consider a complete linear lattice $(L, \preceq)$, in which the bottom and the top elements exist and they are denoted as $0, 1$, respectively. Given a set V, the ordering $\preceq$ in the lattice induces a partial order on the set of L-fuzzy subsets of V, L^V. This ordering provides to L^V the structure of a complete lattice.

A general residuated operator will also be used in this paper to define the fuzzy relation equation, as in [8]. This residuated operator will be denoted as $\odot\colon L \times L \to L$, which is order preserving in both arguments and there exists another operator $\to\colon L \times L \to L$, satisfying the following adjoint property with the conjunctor $\odot$

$$x \odot y \preceq z \quad \text{if and only if} \quad y \preceq x \to z \tag{1}$$

for each $x, y, z \in L$. This property is equivalent to say that $\odot$ preserves supremums in the second argument; $x \odot \bigvee\{y \mid y \in Y\} = \bigvee\{x \odot y \mid y \in Y\}$, for all $Y \subseteq L$.

These operators, as were noted in [8], generalize other kind of residuated pairs [7], [5], since only the monotonicity and the adjoint property are considered.

Definition 1. *Given the pair* $(\odot, \to)$, *a* fuzzy relation equation *is the equation:*

$$R \circ X = T, \tag{2}$$

where $R\colon U \times V \to L$, $T\colon U \times W \to L$ *are given finite* L-*fuzzy relations and* $X\colon V \times W \to L$ *is unknown; and* $R \circ X\colon U \times W \to L$ *is defined, for each* $u \in U$, $w \in W$, *as*

$$(R \circ X)\langle u, w\rangle = \bigvee\{R\langle u, v\rangle \odot X\langle v, w\rangle \mid v \in V\}.$$

It is well known that the fuzzy relation equation (2) has a solution if and only if

$$(R \Rightarrow T)\langle v, w\rangle = \bigwedge\{R\langle u, v\rangle \to T\langle u, w\rangle \mid u \in U\}$$

is a solution and, in that case, it is the greatest solution, see [7], [11], [14].

III. COMPUTING MINIMAL SOLUTIONS ON LINEAR LATTICES

Definition 2. *Given an operator* $\odot\colon L \times L \to L$, *we will say that it holds the* IPNE-condition *(making reference to that* $\odot$ *is Infimum Preserving of arbitrary Non-Empty sets), if it verify*

$$a \odot \bigwedge B = \bigwedge\{a \odot b \mid b \in B\} \tag{3}$$

for each element $a \in L$ *and each non-empty subset* $B \subseteq L$.

From now on, let us consider a general solvable fuzzy relation equation (2), where R, X, T are finite and $\odot$ satisfies the IPNE-condition.

First of all, the auxiliary sets V_{uw} need to be introduced, which are associated with the elements $u \in U$, $w \in W$ and the greatest solution $R \Rightarrow T$. Since for each $u \in U$, $w \in W$

$$\bigvee \{R\langle u, v\rangle \odot (R \Rightarrow T)\langle v, w\rangle \mid v \in V\} = T\langle u, w\rangle, \qquad (4)$$

L is linear and V is finite, there exists at least one $v_s \in V$ validating the equation

$$R\langle u, v_s\rangle \odot (R \Rightarrow T)\langle v_s, w\rangle = T\langle u, w\rangle. \qquad (5)$$

Therefore, the set

$$V_{uw} = \{v \in V \mid R\langle u, v\rangle \odot (R \Rightarrow T)\langle v, w\rangle = T\langle u, w\rangle\}$$

is not empty and, for all $v \notin V_{uw}$, the strict inequality $R\langle u, v\rangle \odot (R \Rightarrow T)\langle v, w\rangle < T\langle u, w\rangle$ holds.

Each v_s in V_{uw} will provide a fuzzy subset S_{uws} as follows: Given $v_s \in V_{uw}$, we have that

$$\{d \in L \mid R\langle u, v_s\rangle \odot d = T\langle u, w\rangle\} \neq \varnothing$$

and the infimum $\bigwedge \{d \in L \mid R\langle u, v_s\rangle \odot d = T\langle u, w\rangle\} = e_s$ also satisfies the equality

$$R\langle u, v_s\rangle \odot e_s = T\langle u, w\rangle$$

by the IPNE-condition. These elements are used to define the fuzzy subsets of V, $Z_{uws} \colon V \to L$, defined by

$$Z_{uws}(v) = \begin{cases} e_s & \text{if } \ v = v_s \\ 0 & \text{otherwise} \end{cases}$$

which form the set Z_{uw}, that is $Z_{uw} = \{Z_{uws} \mid v_s \in V_{uw}\}$, for each $u \in U$, $w \in W$. These sets will be used to characterize the set of solutions of Equation (2) by the notion of *covering*.

Theorem 3. *The L-fuzzy relation $X \colon V \times W \to L$ is a solution of a solvable Equation (2) if and only if $X \preceq (R \Rightarrow T)$ and, for each $w \in W$, the fuzzy subset $X_w \colon V \to L$, defined by $X_w(v) = X\langle v, w\rangle$, is a cover of $\{Z_{uw} \mid u \in U\}$.*

As a consequence, the minimal solutions are characterized by the minimal covers.

Corollary 4. *$X \colon V \times W \to L$ is a minimal solution of Equation (2) if and only if, for each $w \in W$, $X_w \colon V \to L$, defined by $X_w(v) = X\langle v, w\rangle$, is a minimal cover of $\{Z_{uw} \mid u \in U\}$.*

Hence, from the corollary above, minimal solutions of the fuzzy relation equation (2) are obtained from $R \Rightarrow T$. Next, the detailed algorithms are introduced.

Module *MINIMAL_COVERING* uses an usual algorithm in order to compute minimal covering of subsets.

Example III.1. *Let us assume the standard MV–algebra [9], that is, $L = [0, 1]$ is the unit interval, $\odot \colon L \times L \to L$ is the Łukasiewicz operator defined by $x \odot y = \max\{0, x + y - 1\}$ and $\to \colon L \times L \to L$ its residuated implication, defined by $y \to z = \min\{1, 1 - y + z\}$, for all $x, y, z \in [0, 1]$.*

```
input : Universes $U$, $V$ and $W$, the fuzzy relations
        $R\colon U \times V \to L$ and $T\colon U \times W \to L$
output: $MSS$= Set of minimal solutions of the fuzzy
        relation equation $R \circ X = T$
$MSS := [\ ]$;
$S := R \Rightarrow T$, which is the greatest solution of
  $R \circ X = T$ ;
for $k \leftarrow 1$ to $|U|$ and $j \leftarrow 1$ to $|W|$ do
    $Z_{kj} := [\ ]$;
    for $i \leftarrow 1$ to $|V|$ do
        $Z_{kji}$:= zeros row of $|V|$-order;
        if $R[k, i] \odot S[i, j] = T[k, j]$ then
            $e_i := \min\{y \in [0, 1] \mid R[k, i] \,\&\, y = T[k, j]\}$;
            update $Z_{kji}[i]$ by the value $e_i$;
            add $Z_{kji}$ to the list $Z_{kj}$;
        end
    end
end
for $j \leftarrow 1$ to $|W|$ do
    $Z_j := [Z_{1j}, \ldots, Z_{|U|j}]$;
    $[X_j^1, \ldots, X_j^{l_j}]$ :=MINIMAL_COVERING($Z_j$);
end
for $h_1 \leftarrow 1$ to $l_1$ and $\ldots h_{|W|} \leftarrow 1$ to $l_{|W|}$ do
    $X_{h_1 \ldots h_{|W|}}$ := zeros matrix of $|V| \times |W|$-order;
    for $j \leftarrow 1$ to $|W|$ and $i \leftarrow 1$ to $|V|$ do
        $X_{h_1 \ldots h_{|W|}}[i, j] := X_j^{h_j}[i]$ ;
    end
    add $X_{h_1 \ldots h_{|W|}}$ to the list $MSS$;
end
```

Algorithm 1: PMINSOLUTIONS(R, T)

Given $U = \{u_1, u_2, u_3\}$, $V = \{v_1, v_2, v_3\}$ $W = \{w_1, w_2, w_3\}$ and the fuzzy relation equations, defined from the following tables

R	v_1	v_2	v_3
u_1	0.9	0.5	0.9
u_2	0.2	0.9	0.7
u_3	0.8	0.6	0.9

and

T	w_1	w_2	w_3
u_1	0.8	0.4	0.7
u_2	0.6	0.7	0.3
u_3	0.8	0.4	0.6

direct computation shows that the relation $R \Rightarrow T$, defined from the table

$R \Rightarrow T$	w_1	w_2	w_3
v_1	0.9	0.5	0.8
v_2	0.7	0.8	0.4
v_3	0.9	0.5	0.6

is the greatest solution of Equation (2). During the verification we go through the following calculations:
When computing $(R \circ (R \Rightarrow T))\langle u_1, w_1\rangle = 0.8$, we consider the maximum of

$$\begin{array}{lcccc} R\langle u_1, v_1\rangle \odot (R \Rightarrow T)\langle v_1, w_1\rangle & = & 0.9 + 0.9 - 1 & = & 0.8 \\ R\langle u_1, v_2\rangle \odot (R \Rightarrow T)\langle v_2, w_1\rangle & = & 0.5 + 0.7 - 1 & = & 0.2 \\ R\langle u_1, v_3\rangle \odot (R \Rightarrow T)\langle v_3, w_1\rangle & = & 0.9 + 0.9 - 1 & = & 0.8 \end{array}$$

Notice that from v_1 and v_3 we get the maximum. Hence, in order to obtain this maximum, we only need to consider $\{v_1\}$ or $\{v_3\}$. Moreover, the values 0.9 associated with v_1 and 0.9 associated with v_3 cannot be decreased because, if we decrease them, a value less than 0.8 will be obtained in the computation and we do not reach a solution. Therefore, the first column of a solution of Equation (2) could be any column in the set:

$$Z_{1,1} = \{ \begin{pmatrix} 0.9 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0.9 \end{pmatrix} \}$$

However, we need to verify that the other two equalities also hold. Consequently, the equality $(R \circ (R \Rightarrow T))\langle u_2, w_1\rangle = 0.6$ is studied similarly to the previous procedure. The value $(R \circ (R \Rightarrow T))\langle u_2, w_1\rangle$ is the maximum of the values

$$\begin{array}{lclcl}
R\langle u_2, v_1\rangle \odot (R \Rightarrow T)\langle v_1, w_1\rangle & = & 0.2 + 0.9 - 1 & = & 0.1 \\
R\langle u_2, v_2\rangle \odot (R \Rightarrow T)\langle v_2, w_1\rangle & = & 0.9 + 0.7 - 1 & = & 0.6 \\
R\langle u_2, v_3\rangle \odot (R \Rightarrow T)\langle v_3, w_1\rangle & = & 0.7 + 0.9 - 1 & = & 0.6
\end{array}$$

for which $\{v_2\}$ or $\{v_3\}$ is only necessary and so, the first column of a solution of Equation (2) could be one element of the set:

$$Z_{2,1} = \{ \begin{pmatrix} 0 \\ 0.7 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0.9 \end{pmatrix} \}$$

Finally, when computing $(R \circ (R \Rightarrow T))\langle u_3, w_1\rangle = 0.8$ we pass by

$$\begin{array}{lclcl}
R\langle u_3, v_1\rangle \odot (R \Rightarrow T)\langle v_1, w_1\rangle & = & 0.8 + 0.9 - 1 & = & 0.7 \\
R\langle u_3, v_2\rangle \odot (R \Rightarrow T)\langle v_2, w_1\rangle & = & 0.6 + 0.7 - 1 & = & 0.3 \\
R\langle u_3, v_3\rangle \odot (R \Rightarrow T)\langle v_3, w_1\rangle & = & 0.9 + 0.9 - 1 & = & 0.8
\end{array}$$

In this case, only v_3 is necessary and one column is only considered:

$$Z_{3,1} = \{ \begin{pmatrix} 0 \\ 0 \\ 0.9 \end{pmatrix} \}$$

We observe that

$$K = \begin{pmatrix} 0 \\ 0 \\ 0.9 \end{pmatrix} \in Z_{1,1} \cap Z_{2,1} \cap Z_{3,1},$$

so K is the only minimal column which, in an intuitive sense, covers the set $Z_1 = \{Z_{1,1}, Z_{2,1}, Z_{3,1}\}$. Moreover, we conclude that a fuzzy relation X_1, defined as

X_1	w_1	w_2	w_3
v_1	0	0.5	0.8
v_2	0	0.8	0.4
v_3	0.9	0.5	0.6

solves the fuzzy relation equation (2).

Next, we consider the second column of $R \Rightarrow T$, which provides a different case. For $(R \circ (R \Rightarrow T))\langle u_1, w_2\rangle = 0.4$ we have

$$\begin{array}{lclcl}
R\langle u_1, v_1\rangle \odot (R \Rightarrow T)\langle v_1, w_2\rangle & = & 0.9 + 0.5 - 1 & = & 0.4 \\
R\langle u_1, v_2\rangle \odot (R \Rightarrow T)\langle v_2, w_2\rangle & = & 0.5 + 0.8 - 1 & = & 0.3 \\
R\langle u_1, v_3\rangle \odot (R \Rightarrow T)\langle v_3, w_2\rangle & = & 0.9 + 0.5 - 1 & = & 0.4
\end{array}$$

Hence, the maximum is obtained from v_1 or v_3 and, therefore, the following set is considered:

$$Z_{1,2} = \{ \begin{pmatrix} 0.5 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0.5 \end{pmatrix} \}$$

For $(R \circ (R \Rightarrow T))\langle u_2, w_2\rangle = 0.7$ we have

$$\begin{array}{lclcl}
R\langle u_2, v_1\rangle \odot (R \Rightarrow T)\langle v_1, w_2\rangle & = & 0 & & \\
R\langle u_2, v_2\rangle \odot (R \Rightarrow T)\langle v_2, w_2\rangle & = & 0.9 + 0.8 - 1 & = & 0.7 \\
R\langle u_2, v_3\rangle \odot (R \Rightarrow T)\langle v_3, w_2\rangle & = & 0.7 + 0.5 - 1 & = & 0.2
\end{array}$$

Consequently, the subset obtained is

$$Z_{2,2} = \{ \begin{pmatrix} 0 \\ 0.8 \\ 0 \end{pmatrix} \}$$

For $(R \circ (R \Rightarrow T))\langle u_3, w_2\rangle = 0.4$ we have

$$\begin{array}{lclcl}
R\langle u_3, v_1\rangle \odot (R \Rightarrow T)\langle v_1, w_2\rangle & = & 0.8 + 0.5 - 1 & = & 0.3 \\
R\langle u_3, v_2\rangle \odot (R \Rightarrow T)\langle v_2, w_2\rangle & = & 0.6 + 0.8 - 1 & = & 0.4 \\
R\langle u_3, v_3\rangle \odot (R \Rightarrow T)\langle v_3, w_2\rangle & = & 0.9 + 0.5 - 1 & = & 0.4
\end{array}$$

Hence, the assumed subset of columns is

$$Z_{3,2} = \{ \begin{pmatrix} 0 \\ 0 \\ 0.5 \end{pmatrix}, \begin{pmatrix} 0 \\ 0.8 \\ 0 \end{pmatrix} \}$$

In this case, we observe that $Z_{1,2} \cap Z_{2,2} \cap Z_{3,2} = \varnothing$. However,

$$\begin{pmatrix} 0 \\ 0 \\ 0.5 \end{pmatrix}, \begin{pmatrix} 0 \\ 0.8 \\ 0 \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0.8 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0.8 \\ 0 \end{pmatrix} \vee \begin{pmatrix} 0 \\ 0 \\ 0.5 \end{pmatrix},$$

$$\begin{pmatrix} 0.5 \\ 0 \\ 0. \end{pmatrix}, \begin{pmatrix} 0 \\ 0.8 \\ 0 \end{pmatrix} \leq \begin{pmatrix} 0.5 \\ 0.8 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0.8 \\ 0 \end{pmatrix} \vee \begin{pmatrix} 0.5 \\ 0 \\ 0 \end{pmatrix}$$

and $\begin{pmatrix} 0 \\ 0.8 \\ 0.5 \end{pmatrix}, \begin{pmatrix} 0.5 \\ 0.8 \\ 0 \end{pmatrix}$ are the only minimal columns which, again in an intuitive sense, cover the set $Z_2 = \{Z_{1,2}, Z_{2,2}, Z_{3,2}\}$. Moreover, we conclude that the fuzzy relations X_2 and X_3, defined as

X_2	w_1	w_2	w_3
v_1	0	0	0.8
v_2	0	0.8	0.4
v_3	0.9	0.5	0.6

X_3	w_1	w_2	w_3
v_1	0	0.5	0.8
v_2	0	0.8	0.4
v_3	0.9	0	0.6

solve the fuzzy relation (2). Finally, the values in the third column of $R \Rightarrow T$ are reduced.

For $(R \circ (R \Rightarrow T))\langle u_1, w_3\rangle = 0.7$, we compute

$$\begin{array}{lclcl}
R\langle u_1, v_1\rangle \odot (R \Rightarrow T)\langle v_1, w_3\rangle & = & 0.9 + 0.8 - 1 & = & 0.7 \\
R\langle u_1, v_2\rangle \odot (R \Rightarrow T)\langle v_2, w_3\rangle & = & 0 & & \\
R\langle u_1, v_3\rangle \odot (R \Rightarrow T)\langle v_3, w_3\rangle & = & 0.9 + 0.6 - 1 & = & 0.5
\end{array}$$

Hence, $Z_{1,3} = \{ \begin{pmatrix}0.8\\0\\0\end{pmatrix} \}$.

For $(R\circ(R\Rightarrow T))\langle u_2,w_3\rangle = 0.3$*, we have*

$$\begin{array}{rcccl} R\langle u_2,v_1\rangle \odot (R\Rightarrow T)\langle v_1,w_3\rangle &=& 0.2+0.8-1 &=& 0\\ R\langle u_2,v_2\rangle \odot (R\Rightarrow T)\langle v_2,w_3\rangle &=& 0.9+0.4-1 &=& 0.3\\ R\langle u_2,v_3\rangle \odot (R\Rightarrow T)\langle v_3,w_3\rangle &=& 0.7+0.6-1 &=& 0.3 \end{array}$$

two possibilities providing two columns: $Z_{2,3} = \{ \begin{pmatrix}0\\0.4\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0.6\end{pmatrix} \}$.

For $(R\circ(R\Rightarrow T))\langle u_3,w_3\rangle = 0.6$ *we have*

$$\begin{array}{rcccl} R\langle u_3,v_1\rangle \odot (R\Rightarrow T)\langle v_1,w_3\rangle &=& 0.8+0.8-1 &=& 0.6\\ R\langle u_3,v_2\rangle \odot (R\Rightarrow T)\langle v_2,w_3\rangle &=& 0.6+0.4-1 &=& 0\\ R\langle u_3,v_3\rangle \odot (R\Rightarrow T)\langle v_3,w_3\rangle &=& 0.9+0.6-1 &=& 0.5 \end{array}$$

Therefore, $Z_{3,3} = \{ \begin{pmatrix}0.8\\0\\0\end{pmatrix} \}$.

In this case, there are two minimal covering of the set $Z_3 = \{Z_{1,3}, Z_{2,3}, Z_{3,3}\}$*:*

$$\begin{pmatrix}0.8\\0.4\\0\end{pmatrix} = \begin{pmatrix}0.8\\0\\0\end{pmatrix} \vee \begin{pmatrix}0\\0.4\\0\end{pmatrix} \quad \textit{and} \quad \begin{pmatrix}0.8\\0\\0.6\end{pmatrix} = \begin{pmatrix}0.8\\0\\0\end{pmatrix} \vee \begin{pmatrix}0\\0\\0.6\end{pmatrix}$$

This yields four fuzzy relations, defined as follows

X_4	w_1	w_2	w_3
v_1	0	0	0.8
v_2	0	0.8	0.4
v_3	0.9	0.5	0

X_5	w_1	w_2	w_3
v_1	0	0	0.8
v_2	0	0.8	0
v_3	0.9	0.5	0.6

X_6	w_1	w_2	w_3
v_1	0	0.5	0.8
v_2	0	0.8	0.4
v_3	0.9	0	0

X_7	w_1	w_2	w_3
v_1	0	0.5	0.8
v_2	0	0.8	0
v_3	0.9	0	0.6

that solve Equation (2). By their construction and the properties of the Łukasiewicz conjunctor, they are minimal solutions.

Example III.2. *In this example, we consider the Gödel structure [9], then* $L = [0,1]$ *and* $\odot\colon L\times L\to L$ *and* $\to\colon L\times L\to L$ *are defined by* $x\odot y = \min\{x,y\}$ *and*

$$y \to z = \begin{cases} 1 & \textit{if } y\le z\\ z & \textit{otherwise} \end{cases}$$

for all $x,y,z\in[0,1]$*. Given* $U=\{u_1,u_2\}$*,* $V=\{v_1,v_2,v_3\}$*,* $W=\{w\}$ *and*

R	v_1	v_2	v_3
u_1	0.6	0.4	0.5
u_2	0.8	0.7	0.6
u_3	0.9	1	0.9

T	w
u_1	0.6
u_2	0.7
u_3	0.9

the direct computation shows that

$R\Rightarrow T$	w
v_1	0.7
v_2	0.9
v_3	1.0

is the maximal solution of Equation (2). In order to verify the equality $(R\circ(R\Rightarrow T))\langle u_1,w\rangle = 0.6$ *we compute*

$$\begin{array}{rcccl} R\langle u_1,v_1\rangle \odot (R\Rightarrow T)\langle v_1,w\rangle &=& 0.6\wedge 0.7 &=& 0.6\\ R\langle u_1,v_2\rangle \odot (R\Rightarrow T)\langle v_2,w\rangle &=& 0.4\wedge 0.9 &=& 0.4\\ R\langle u_1,v_3\rangle \odot (R\Rightarrow T)\langle v_2,w\rangle &=& 0.5\wedge 1.0 &=& 0.5 \end{array}$$

Note that we only need the value associated with v_1*. Moreover, this value can be reduced until* 0.6*. Hence, the first (and only) column of a solution has to be contained in the following set*

$$Z_{1,1} = \{ \begin{pmatrix}x\\0\\0\end{pmatrix} \mid 0.6\le x\le 1\}$$

Focusing on our main goal, the least one is the column associated with a minimal solution. Hence, we only consider the column $Z_{1,1} = \{ \begin{pmatrix}0.6\\0\\0\end{pmatrix} \}$

For $(R\circ(R\Rightarrow T))\langle u_2,w\rangle = 0.7$ *we have*

$$\begin{array}{rcccl} R\langle u_2,v_1\rangle \odot (R\Rightarrow T)\langle v_1,w\rangle &=& 0.8\wedge 0.7 &=& 0.7\\ R\langle u_2,v_2\rangle \odot (R\Rightarrow T)\langle v_2,w\rangle &=& 0.7\wedge 0.9 &=& 0.7\\ R\langle u_2,v_3\rangle \odot (R\Rightarrow T)\langle v_3,w\rangle &=& 0.6\wedge 1.0 &=& 0.6 \end{array}$$

Now the values associated with v_1 *and* v_2 *provide the maximum. Furthermore, the value for* v_2 *can also be decreased, specifically, any element* x *in* $[0.7,1]$ *provides the same maximum result:* $0.7\wedge x = 0.7$*. Therefore, focusing on the minimal solutions we only need to consider:*

$$Z_{2,1} = \{ \begin{pmatrix}0.7\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0.7\\0\end{pmatrix} \}$$

Finally, for $(R\circ(R\Rightarrow T))\langle u_3,w\rangle = 0.9$ *we calculate*

$$\begin{array}{rcccl} R\langle u_3,v_1\rangle \odot (R\Rightarrow T)\langle v_1,w\rangle &=& 0.9\wedge 0.7 &=& 0.7\\ R\langle u_3,v_2\rangle \odot (R\Rightarrow T)\langle v_2,w\rangle &=& 1.0\wedge 0.9 &=& 0.9\\ R\langle u_3,v_3\rangle \odot (R\Rightarrow T)\langle v_3,w\rangle &=& 0.9\wedge 1.0 &=& 0.9 \end{array}$$

In this last case v_2 *and* v_3 *are involved in the computation of the maximum and the value associated with* v_3 *can be decreased until* 0.9*. These considerations yield the following minimal solutions*

$$\begin{pmatrix}0.6\\0.7\\0.9\end{pmatrix}, \begin{pmatrix}0.7\\0\\0.9\end{pmatrix}, \begin{pmatrix}0.6\\0.9\\0\end{pmatrix}$$

IV. CONCLUSION AND FUTURE WORKS

The main aim of this research is to define as generally as possible an algebraic structure that allows the existence of minimal solutions of the fuzzy relation equations defined based on this structure. For that, a general increasing operation $\odot$, which only satisfies the adjointness property, i.e. is residuated, and satisfies the IPNE-condition, has been considered to define a general fuzzy relation equation, which has minimal solutions whenever a solution exists. Moreover, a new algebraic characterization using the notion of covering is introduced, which provides a method to obtain the minimal solutions and, consequently, the whole set of solutions.

As future work, the obtained results will be applied to several problems in fuzzy logic, such as to abduction reasoning. It is well-known that implications in MV-algebras are infinitely distributive. A topic of future study is to characterize all structures where implication is infinitely distributivity. Algebraic structures that satisfy the INPE-condition are not studied much; also they will be a topic of future research.

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