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A Numerical Algorithm of Solving the Forced sine-Gordon Equation

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***Abstract*—The numerical method of solving the problem of small perturbations of a stationary traveling solution (soliton) of well-known in physics sin-Gordon equation is presented. The solution is reduced to solving a set of linear hyperbolic partial differential equations. The Riemann function method is used to find a solution of a linear PDE. The value of the Riemann function at any particular point is found as a solution of an ordinary differential equation. An algorithm of calculation of a double integral over a triangular integration area is given.**

I. Introduction

The RIEMANN method is an important technique for solving Cauchy problems for partial differential equations. However, it does not yield closed form solutions except in few cases of equations with constant coefficients [1]. A typical example of such an equation is the forced sine-Gordon equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \sin u = \varepsilon\Phi(t) \tag{1}$$

where $\varepsilon = 1$ is a small parameter, t is a time variable, x is a space variable, and $\Phi(t)$ is a deterministic or random external force with known statistics. Equation (1) with zero right-hand side possesses stationary traveling solutions depending on a variable $\chi = x - Vt$ $(-1 < V < 1)$. The particular solution that is of great physical interest is the kink or soliton [2] (Fig. 1)

$$u_0(\chi) = 4\text{atan}\left[\exp\left(\chi/\sqrt{a}\right)\right] \tag{2}$$

where $a = 1 - V^2$. The problem of deterministic or stochastic perturbations of the kink solution is important in physical applications.

The approximate solution of the equation (1) with non-zero function $\Phi(t)$ can be constructed by the asymptotic method using a smallness of the parameter ε:

$$u(\chi,t) = u_0(\chi) + \varepsilon u_1(\chi,t) + \varepsilon^2 u_2(\chi,t) + \text{K} \tag{3}$$

where $u_0(\chi)$ is the solution (2). The functions $u_1, u_2, ...$ are solutions of linear hyperbolic equations.

Apparently, Walsh's papers [3], [4] were the most remarkable first investigation of stochastic processes described by the second order partial differential equations. In particular, a weak solution of a stochastic partial differential equation was defined as a solution of an integral equation. Walsh in [2] applied the Green's method to simple linear hyperbolic and parabolic equations in case of the white noise, and Carmona and Nualart in [5] proved that the weak solution exists and it is unique.

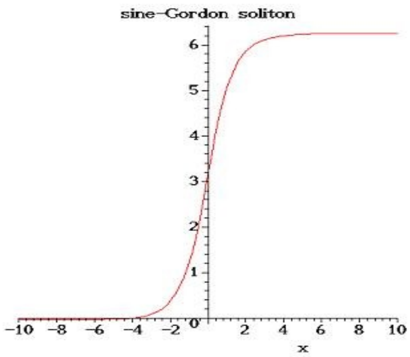

Fig. 1. The sin-Gordon soliton.

In this paper, the Riemann function method is used to find a solution of linear hyperbolic equations (4) and (5). The algorithm is numerical and can be divided into two parts. First, the Riemann function at each point of the integration area is found as a solution of an ordinary differential equation. Second, the triangular integration area is transformed into a rectangle. This allows simplifying and improving considerably calculation of a double integral. Initially the method was described in [6] and [7] and was applied in a case of stochastic perturbations. In the current paper, the proposed method has been improved. The physical part of the method and some numerical results were described in [8].

II. The Solution Form of the Eequation (1)

In the new variables (χ,t) the equation (1) becomes

$$\frac{\partial^2 u}{\partial t^2} - 2V\frac{\partial^2 u}{\partial\chi\partial t} - a\frac{\partial^2 u}{\partial\chi^2} + \sin u = \varepsilon\Phi(t) \tag{4}$$

Expanding $\sin u$ in power series in ε

$$\sin u(\chi,t) = \sin u_0(\chi) + \varepsilon u_1(\chi,t)\cos u_0(\chi) + \varepsilon^2\left[u_2(\chi,t)\cos u_0(\chi) - \frac{u_1^2(\chi,t)}{2}\sin u_0(\chi)\right] + \ldots \quad (5)$$

Substituting (3) and (5) into (4) in the first order on ε we obtain

$$\frac{\partial^2 u_1}{\partial t^2} - 2V\frac{\partial^2 u_1}{\partial\chi\partial t} - a\frac{\partial^2 u_1}{\partial\chi^2} + \cos u_0(\chi)u_1 = \Phi(t) \quad (6)$$

In the second order

$$\frac{\partial^2 u_2}{\partial t^2} - 2V\frac{\partial^2 u_2}{\partial\chi\partial t} - (1-V^2)\frac{\partial^2 u_2}{\partial\chi^2} + \cos u_0(\chi)u_2 = \frac{1}{2}u_1^2\sin u_0(\chi) \quad (7)$$

To apply the Riemann's method of solving (6) and (7) [1] we need to transform these equations to the standard form which does not contain the second mixed derivative. To get rid of the mixed derivatives let us make a transformation $\chi = \chi$ and $\tau = t - \frac{V\chi}{a}$. In the new variables equations (6) and (7) read

$$\frac{\partial^2 u_1}{\partial\tau^2} - a^2\frac{\partial^2 u_1}{\partial\chi^2} + a\cos u_0(\chi)u_1 = a\sin\left(\tau + \frac{V\chi}{a}\right) \quad (8)$$

and

$$\frac{\partial^2 u_2}{\partial\tau^2} - a^2\frac{\partial^2 u_2}{\partial\chi^2} + a\cos u_0(\chi)u_2 = \frac{1}{2}au_1^2\sin u_0(\chi) \quad (9)$$

with trivial initial conditions over the straight line C

$$C:\tau = -\frac{V}{a}\xi \quad (10)$$

III. The Riemann Function

Equations (8) and (9) have the same left-hand side and can be presented as

$$\frac{\partial^2 u_1}{\partial\tau^2} - a^2\frac{\partial^2 u_1}{\partial\chi^2} + a[1-f(\chi)]u_1 = a\Phi\left(\tau + \frac{V\chi}{a}\right) \quad (11)$$

$$\frac{\partial^2 u_2}{\partial\tau^2} - a^2\frac{\partial^2 u_2}{\partial\chi^2} + a[1-f(\chi)]u_2 = \frac{a}{2}u_1^2\sin u_0(\chi) \quad (12)$$

where $f(\chi) = 2/\cosh^2(\chi/\sqrt{a})$ since

$$\cos[4\arctan(e^{\frac{1}{\sqrt{a}}\chi})] = 1 - \frac{2}{\cosh^2(\frac{1}{\sqrt{a}}\chi)}$$

The graph of the function $f(\chi)$ for the value $V = 0.5$ is shown in Fig. 2.

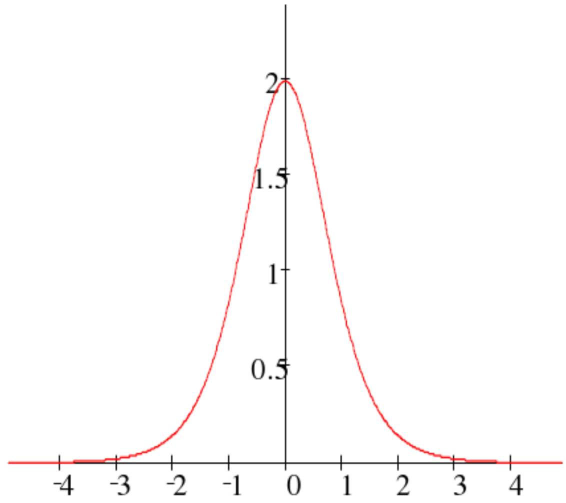

Fig .2. The function $f(\chi)$, $a = 1$.

The corrections $u_i(\chi_0,\tau_0), i = 1,2$ to the kink solution $u_0(\chi_0)$ at a point (χ_0,τ_0) can be presented through the Riemann function $\omega(\chi,\tau)$ [1]:

$$u_1(\chi_0,\tau_0) = a\iint_A \Phi\left(\tau + \frac{V\chi}{a}\right)\omega(\chi,\tau)d\chi d\tau \quad (13)$$

$$u_2(\chi_0,\tau_0) = \frac{a}{2}\iint_A u_1^2(\chi,\tau)\sin u_0(\chi)\,\omega(\chi,\tau)d\chi d\tau$$

(14)

where A is the characteristic triangle in the plane (χ,τ), bounded by the straight line C given in (10) and the characteristics $\partial A_{\pm}:\tau \;\; \frac{1}{a}(\chi-\chi_0) = \tau_0$ (Fig. 3).

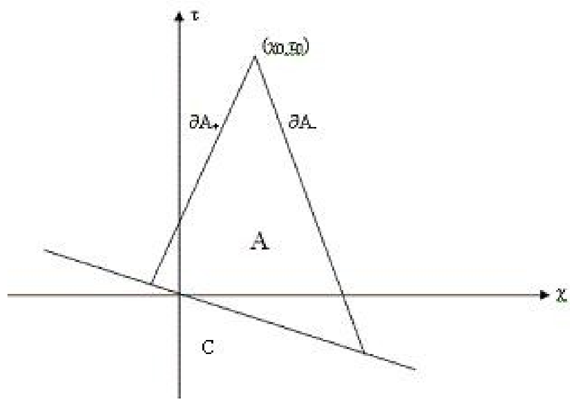

Fig. 3. The characteristic triangle A.

The Riemann function $\omega(\chi,\tau)$ satisfies the equation

$$\frac{\partial^2\omega}{\partial\tau^2} - a^2\frac{\partial^2\omega}{\partial\chi^2} + b^2(1-f(\chi))\omega = 0 \quad (15)$$

with $\omega = 1/(2a)$ over the characteristics and $b = \sqrt{a}$.

Since $f(\chi) \to 0$ outside of a finite interval $(-\chi_{\lim}, \chi_{\lim})$ then one can expect that the solution of (15) can be close to the solution of the telegrapher's equation [1]:

$$\frac{\partial^2 \omega}{\partial \tau^2} - a^2 \frac{\partial^2 \omega}{\partial \chi^2} + b^2 \omega = 0$$

which is $\omega(\chi,\tau) = \frac{1}{2a} J_0(b\gamma)$ and

$\gamma = \sqrt{(\tau-\tau_0)^2 - \frac{1}{a^2}(\chi-\chi_0)^2}$. The variable γ is a hyperbolic distance in the characteristic triangle A . The distance γ is positive inside A, i.e. $(\tau-\tau_0)^2 > \frac{1}{a^2}(\chi-\chi_0)^2$, and imaginary when $(\tau-\tau_0)^2 < \frac{1}{a^2}(\chi-\chi_0)^2$. It vanishes when $(\tau-\tau_0)^2 = \frac{1}{a^2}(\chi-\chi_0)^2$.

The equation (15) in the variables (χ,γ) has the form

$$\frac{\partial^2 \omega}{\partial \gamma^2} - a^2 \frac{\partial^2 \omega}{\partial \chi^2} + 2\frac{\chi-\chi_0}{\gamma}\frac{\partial^2 \omega}{\partial \chi \partial \gamma} + \frac{1}{\gamma}\frac{\partial \omega}{\partial \gamma} + b^2(1-f(\chi))\omega = 0 \qquad (16)$$

Let's look for the solution of (16) in the form

$$\omega(\chi,\gamma) = \frac{1}{2a} J_0(b\gamma) + \Phi(\chi)\sum_{n=1}^{\infty} A_n (b\gamma)^n \qquad (17)$$

where the first term of the sum is the solution of the telegrapher's equation and the second term must be small for a finite value of $\chi_{\lim}$.

Since

$$J_0''(b\gamma) + \frac{1}{b\gamma} J_0'(b\gamma) + J_0(b\gamma) = 0$$

then substitution of (17) into (16) gives

$$\Phi(\chi)\sum_{n=1}^{\infty} n(n-1)A_n b^2 (b\gamma)^{n-2} - a^2\Phi''(\chi)\sum_{n=1}^{\infty} A_n (b\gamma)^n +$$
$$2(\chi-\chi_0)\Phi'(\chi)\sum_{n=1}^{\infty} nA_n b^2 (b\gamma)^{n-2} + \Phi(\chi)\sum_{n=1}^{\infty} nA_n b^2 (b\gamma)^{n-2}$$
$$+b^2\Phi(\chi)(1-f(\chi))\sum_{n=1}^{\infty} A_n (b\gamma)^n = \frac{1}{2} f(\chi) J_0(b\gamma)$$

For $\chi = \chi_0$ the last equation becomes

$$A_1\Phi(\chi_0)\frac{1}{b\gamma} + 4A_2\Phi(\chi_0) + \sum_{n=1}^{\infty}[(n+2)^2\Phi(\chi_0)A_{n+2}$$
$$+(-a\Phi''(\chi_0) + (1-f(\chi_0))\Phi(\chi_0)A_n]\gamma^n$$
$$= \frac{1}{2a} f(\chi_0)[1 - \frac{\gamma^2}{2^2} + \frac{\gamma^4}{2^2 4^2} - \frac{\gamma^6}{2^2 4^2 6^2} + \ldots]$$

and, therefore,

$A_n = 0$ *for odd n*,

$$A_2 = \frac{1}{2a}\frac{f(\xi_0)}{\varphi(\xi_0)}\frac{1}{2^2},$$

$$A_{2k+2} = [\frac{f(\xi_0)}{2a}\frac{(-1)^k}{2^2 4^2 \ldots (2k)^2} + (\frac{a^2}{b^2}\varphi''(\xi_0) -$$
$$(1-f(\xi_0))\varphi(\xi_0))A_{2k}]\frac{1}{(2k+2)^2\varphi(\xi_0)},$$

$k = 1,2,3\ldots$

Assume

$$\frac{a^2}{b^2}\Phi''(\chi_0) = (1-f(\chi_0))\Phi(\chi_0), \Phi(\chi_0) = f(\chi_0) \qquad (17)$$

It follows that

$$A_{2k+2} = \frac{1}{2a}\frac{(-1)^k}{2^2 4^2 \ldots (2k)^2}, k = 1,2,3\ldots$$

and

$$\omega(\chi,\gamma) = \frac{1}{2a}[J_0(b\gamma) + (1-J_0(b\gamma))\varphi(\chi)] \qquad (18)$$

Substitute (18) into (16) and we obtain that the solution $\varphi(\chi)$ satisfies the following equation

$$-a(1-J_0(b\gamma))\varphi''(\chi) +$$
$$(J_0(b\gamma)+J_2(b\gamma))(\chi-\chi_0)\varphi'(\chi) + \qquad (19)$$
$$(1-f(\chi)(1-J_0(b\gamma)))\varphi(\chi) = J_0(b\gamma)f(\chi)$$

and subject to the boundary conditions

$$\varphi(\chi_0) = f(\chi_0), \quad f(\pm\infty) = 0 \qquad (20)$$

IV. Numerical Algorithm and Calculations

A solution $\omega(\chi_1,\tau_1)$ of the partial differential equation (15) at any particular point (χ_1,τ_1) can be reduced to solving the boundary value problem (19), (20) with fixed value of γ , where

$$\gamma = \sqrt{(\tau_1-\tau_0)^2 - \frac{1}{a^2}(\chi_1-\chi_0)^2} .$$

A family of curves $\gamma = const$ define hyperbolae imbedded into the characteristic triangle (Fig.4). The solution surface (15) can be represented as a family of curves over the hyperbolae in the 3D space (χ,τ,ω) . The surface representing the Riemann function for $\chi_0 = 0$ is shown in Fig. 5 and was drawn using MAPLE.

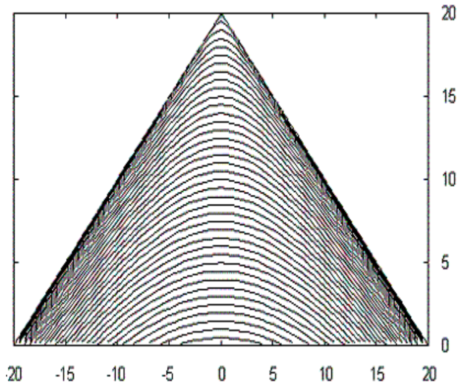

Fig. 4. The Characteristic triangle with the family of curves $\gamma = const$.

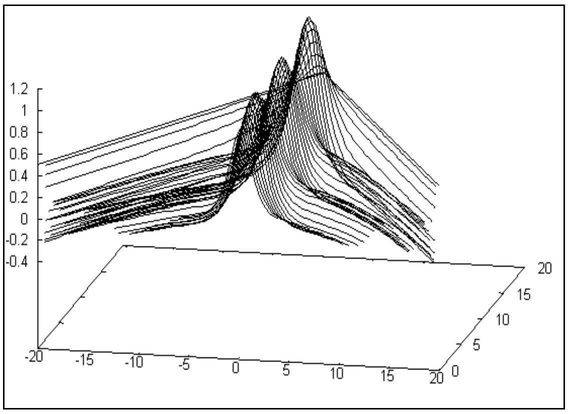

Fig. 5. The solution surface of (15) representing the Riemann function. Each curve is a solution of (19), (20) for particular fixed value of γ .

The boundary value problem (19), (20) can be solved numerically using the relaxation method [9]. Since the solution of this BVP approaches zero when $\chi \to \pm\infty$ then for numerical calculations this problem can be solved for finite values of $\chi_{\lim}$ which are found manually from the condition $\lim_{\chi\to 0} f(\chi) \to 0$.

In the numerical calculations of the double integrals (13) and (14) one of the most difficult tasks is integration over the triangle A. First, the integral needs to be calculated for various values of (χ_0, τ_0). The area of the rectangle A becomes larger when the value of τ_0 increases. The second difficulty is that changing the value of the parameter a leads to changing the rectangle A shape. Therefore, it is very difficult to develop a universal algorithm for various values of (χ_0, τ_0) and the parameter a . The integration area A (Fig. 3) can be mapped into a rectangle $R = \left\{ \frac{-1+V}{1+V} \le v \le 1, 0 \le u \le 1 \right\}$ (Fig. 6) with coordinates (v, u) by means of transformation

$$\chi = \chi_0 + a \cdot t_0 \left(1-u\right) v / \left(1-V\right),$$

$$\tau = t_0 u - V \chi_0 / a - V t_0 \left(1-u\right) v / \left(1-V\right),$$

where $t_0 = \tau_0 + V \chi_0 / a$.

This transformation allows simplifying significantly numerical integration and speed up the algorithm of calculation of functions $u_{1,2}.(\chi, t)$. The area of integration can be covered by a rectangular mesh and then, the standard Simpson method can be used [9].

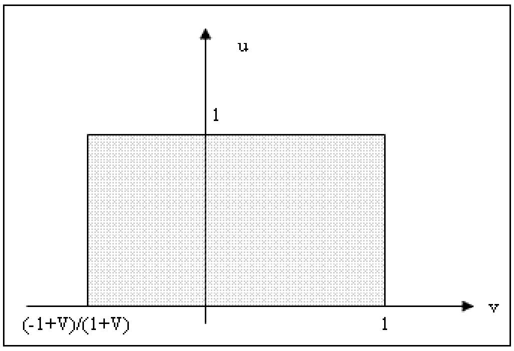

Fig. 6. The rectangular integration area R(v, u).

The first integral (13) in the new variables (v, u) has the form

$$u_1\left(\chi_0, \tau_0\right) = \frac{t_0^2}{2}\left(1+V\right) \iint_R \{(1-u)\Phi(t_0 u) \times$$
$$\left[J_0\left(\alpha(v,u)\right) + \left\{1 - J_0\left(\alpha(v,u)\right)\right\} \varphi(v,u) \right] \} dudv \quad (21)$$

$$\alpha(v,u) = t_0 (1-u) \sqrt{(1+V)(1-v)(1+v+Vv-V)}$$

where the double integral is calculated using the Simpson method and the value of $\varphi(v,u)$ at each point of the integral sum is a solution of the boundary value problem (19), (20).

The numerical algorithm was implemented in a program written in C++. For calculating values of the Bessel functions, double integrals and solving ordinary differential equations the code given in [9] was used.

As an example, the function $\Phi(t) = \sin(10t)$ was considered. Two graphs in a plane (x, u_1) representing results of calculations of the integral (21) for $-1 \le x \le 1$ are shown in Fig. 7 and Fig. 8.

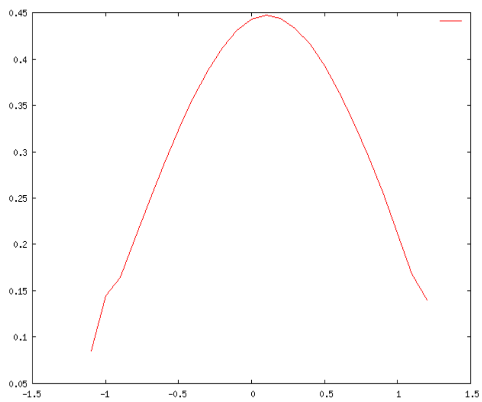

Fig. 7. $V = 0.95, t = \pi$

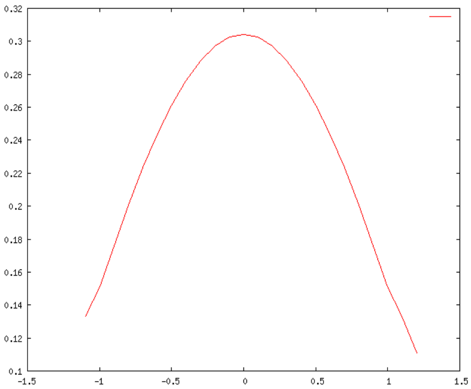

Fig. 8. $V = 0, t = 4088$

The values of ε in (3) depend on a particular physical problem and are not discussed in this paper. They can be up to 0.3 in some cases.

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