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On a class of defuzzification functionals

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***Abstract*—Classical convex fuzzy numbers have many disadvantages. The main one is that every operation on this type of fuzzy numbers induces the growing fuzziness level. Another drawback is that the arithmetic operations defined for them are not complementary, for instance: addition and subtraction. Therefore the first author (W. K.) with his coworkers has proposed the extended model called ordered fuzzy numbers (OFN). The new model overcomes the above mentioned drawbacks and at the same time has the algebra of crisp (non-fuzzy) numbers inside. Ordered fuzzy numbers make possible to utilize the fuzzy arithmetic and to construct the Abelian group of fuzzy numbers and then an algebra. Moreover, in turn out, that four main operations introduced are very suitable for their algorithmisation. The new attitudes demand new defuzzification operators. In the linear case they are described by the well–know functional representation theorem valid in function Banach spaces. The case of nonlinear functionals is more complex, however, it is possible to prove general, uniform approximation formula for nonlinear and continuous functionals in the Banach space of OFN. Counterparts of defuzzification functionals known in the Mamdani approach are also presented, some numerical experimental results are given and conclusions for further research are drawn.**

I. Introduction

CLASSICAL fuzzy numbers are very special fuzzy sets defined on the universe of all real numbers. Fuzzy numbers are of great importance in fuzzy systems. In the applications, the triangular and the trapezoidal fuzzy numbers are usually used.

There are two commonly accepted methods of dealing with fuzzy numbers, both basing on the classical concept of fuzzy sets, namely on the membership functions. The first, more general approach deals with the so-called convex fuzzy numbers of Nguyen [31], while the second one deals with shape functions and $L-R$ numbers, set up by Dubois and Prade [6].

When operating on convex fuzzy numbers we have the interval arithmetic for our disposal. However, the approximations of shape functions and operations are needed, if one wants to remain within the $L-R$ numbers while following the Zadehïż¡s extension principle [38]. In this representation (in most cases) calculation results are not exact and are questionable if some rigorous and exact data are needed, e.g. in the control or modeling problems. This can be treated as a drawback of the properties of classical fuzzy algebraic operations.

In the literature, moreover, it is well know that unexpected and uncontrollable results of repeatedly applied operations, caused by the need of making intermediate approximations (remarked in [34],[35]) can appear. This rises the heavy argument for those who still criticize the fuzzy number calculus. Fortunately, it was already noticed by both Dubois and Prade in their recent publication [8] that something is missing in the definition of the fuzzy numbers and the operations on them.

In most cases one assumes that a typical membership function ïż¡μ_A of a fuzzy number A satisfies convexity assumptions requiring after Nguyen [31] all α-cuts and the support of A to be convex subsets of $\mathbf{R}$. At this stage it seems necessary to recall both notions used: the α-cut of A is a (classical) set $A[\alpha] = \{x \in \mathbf{R} : \mu_A(x) \geq \alpha\}$, for each $\alpha \in [0,1]\}$, and the support of A is the (classical) set $\text{supp}(A) = \{x \in \mathbf{R} : \ \mu_A(x) > 0\}$. One additionally assumes [2], [3], [5], [12], [31], [34] that the convex fuzzy number A has its core, i.e. the (classical) set of those $x \in \mathbf{R}$ for which its membership function $\mu_A(x) = 1$, which is not empty and its support is bounded. Then the arithmetic of fuzzy numbers can be developed using both the Zadeh's extension principle [38], [39] and the α-cut with interval arithmetic method [12].

As long as one works with fuzzy numbers that possess continuous membership functions, the two procedures: the extension principle and the $\alpha - cut$ and interval arithmetic method give the same results (cf. [2]). The results of multiple operations on convex fuzzy numbers are leading to a large growth of the fuzziness, and depend on the order of the operations since the distributive law, which involves the interaction of addition and multiplication, does hold there. Moreover, the use of the extension principle in the definition of the arithmetic operations on fuzzy numbers is generally numerically inefficient. These operations cannot be equipped with a linear structure and hence no norm can be defined on them. Standard algebraic operations on fuzzy numbers basing on the Zadehïż¡s extension principle and those for fuzzy numbers of $L - R$ type or convex fuzzy numbers (see [5]) which are making use of interval analysis have several drawbacks. They are listed in our previous publications [22],[23], [24],[26], [16].

In our opinion the main drawback is the lack of a solution X to the most simple fuzzy arithmetic equation

$$A + X = C \tag{1}$$

with known fuzzy numbers A and C. If the support of C is greater than that of A a unique solution in the form of a fuzzy number X exists. However, this is the only case, since for A with larger support than that of C the solution does not exist. Another drawback is related to the fact that in general $A + B - A$ is not equal to B.

The goal of the authors of the previous papers [22], [23], [24], [25], [27] was to overcome the above mentioned drawbacks by constructing a revised concept of fuzzy number and at the same time to have the algebra of crisp (non-fuzzy) numbers inside the concept.

In our investigations we wanted to omit, to some extend, the arithmetic based on the α–cut of membership functions of fuzzy numbers (sets), and to be close to the operations known from the real line. It was noticed by Dubois and Prade [7] in 2005, (and repeated recently in [9]) after our definition of the *ordered fuzzy number* [26] had been given, that the concept of fuzzy number is too close to the concept of interval. Our new concept makes it possible to utilize the fuzzy arithmetic in a simple way and to construct an Abelian group of fuzzy numbers, and then an algebra. At the same time the new model contains the cone of convex fuzzy numbers. The definition presented here contains all continuous convex fuzzy numbers, however, its recent enlargement presented in [19] includes all convex fuzzy numbers. Moreover, the new model contains more elements and each convex fuzzy number leads to two different new fuzzy numbers, called here the *ordered fuzzy numbers*, which differ by their orientation. This will become more evident later. Additionally, in turns out that the four main operations introduced are very suitable for algorithmisation. We should stress, however, that the arithmetic of the new model restricted to convex (continuous) fuzzy numbers gives different results in comparison to that of the interval arithmetic. This is evident already in the scalar multiplication and subtraction. However, this gives us the chance to solve the arithmetic equation (1) for any pair of fuzzy numbers A and C.

The organization of the paper is as follows. In Section 2 we repeat our main definition and basic properties of extended model of fuzzy numbers presented in the series of papers [15], [16], [21], [22], [23], [24], [25], [26]. Then defuzzification functionals are discussed. First, the linear case, then the nonlinear one. Then a counterpart of the Mamdani center of gravity defuzzification functional is derived. In the final section conclusions together with numerical results of some experiments with implementations of the derived formula are presented.

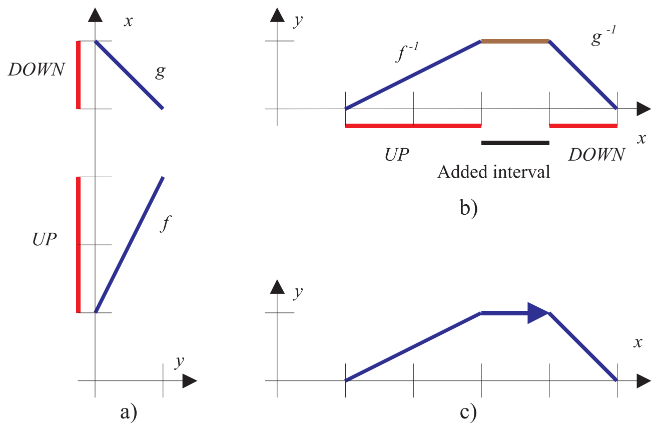

Fig. 1. a) Example of an ordered fuzzy number; b) construction of the membership function; c) the arrow denotes the orientation and the order of inverted functions: first f and then g.

II. Basic Properties of Ordered Fuzzy Numbers

Definition 1. By an *ordered fuzzy number* A we mean an ordered pair (f, g) of functions such that $f, g : [0, 1] \to \mathbf{R}$ are continuous.

Notice that in our definition we do not require that two continuous functions f and g are (partial) inverses of some membership function. Moreover, it may happen that membership function corresponding to A does not exist. We call the corresponding elements: f—the up-part and g—the down-part of the fuzzy number A. To be in agreement with further and classical denotations of fuzzy sets (numbers), the independent variable of the both functions f and g is denoted by y, and their values by x. The continuity of both parts implies their images are bounded intervals, say UP and $DOWN$, respectively (Fig. 1). We have used symbols to mark boundaries for $UP = [l_A, 1_A^-]$ and for $DOWN = [1_A^+, p_A]$. In general, the functions f and g need not to be invertible as functions of y, only continuity is required. If we assume, however, that 1) they are monotonous: f is increasing, and g is decreasing, and such that 2) $f \leq g$ (pointwise), we may define the membership function $\mu(x) = f^{-1}(x)$, if $x \in [f(0), f(1)] = [l_A, 1_A^-]$, and $\mu(x) = g^{-1}(x)$, if $x \in [g(1), g(0)] = [1_A^+, p_A]$ and ïż¡$\mu(x) = 1$ when $x \in [1_A^-, 1_A^+]$. In this way we obtain the membership function ïż¡$\mu(x), x \in \mathbf{R}$. When the functions f and/or g are not invertible or the condition 2) is not satisfied then in the plane $x - y$ the membership curve (or relation) can be defined, composed of the graphs of f and g and the line $y = 1$ over the core $\{x \in [f(1), g(1)]\}$.

Notice that in general $f(1)$ needs not be less than $g(1)$ which means that we can reach improper intervals, which have been already discussed in the framework of the extended interval arithmetic by Kaucher [11]. In such case Prokopowicz has introduced in [33] the *corresponding* membership function which can be defined by the formulae:

$$\mu(x) = \max \arg\{f(y) = x, g(y) = x\} \tag{2}$$
$$\text{if } x \in \text{Range}(f) \cup \text{Range}(g) \ ,$$

$$\mu(x) = 1 \quad \text{if } x \in [f(1), g(1)] \cup [g(1), f(1)], \tag{3}$$
$$\text{and } \mu(x) = 0, \text{ otherwise}, \tag{4}$$

where one of the intervals $[f(1), g(1)]$ or $[g(1), f(1)]$ may be empty, depending on the sign of $f(1) - g(1)$, (i.e., if the sign is -1 then the second interval is empty).

A. Norm and partial order

Let $\mathcal{R}$ be a universe of all OFN's. Notice that this set is composed of all pairs of continuous functions defined on the closed interval $I = [0, 1]$, and is isomorphic to the linear space of real 2D-vector valued functions defined on the unit interval I with the norm of $\mathcal{R}$ as follows

$$||A|| = \max(\sup_{s \in I} |f_A(s)|, \sup_{s \in I} |g_A(s)|) \text{ if } A = (f_A, g_A) \ .$$

The space $\mathcal{R}$ is topologically a Banach space. The neutral element of addition in $\mathcal{R}$ is a pair of constant functions equal to crisp zero. It is also a Banach algebra with unity: the multiplication has a neutral element – the pair of two constant functions equal to one, i.e., the crisp one.

A relation of partial ordering in $\mathcal{R}$ can be introduced by defining the subset of 'positive' ordered fuzzy numbers: a number $A = (f, g)$ is not less than zero, and by writing

$$A \geq 0 \quad \text{iff} \quad f \geq 0, g \geq 0 \ . \tag{5}$$

In this way the set $\mathcal{R}$ becomes a partially ordered ring.

III. Representation of Defuzzification Functional

Defuzzification is a main operation in fuzzy controllers and fuzzy inference systems where fuzzy inference rules appear, in the course of which to a membership function representing classical fuzzy set a real number is attached. We know a number of defuzzification procedures from the literature. Since classical fuzzy numbers are particular case of fuzzy sets the same problem appears when rule's consequent part is a fuzzy number. Then the problem arises what can be done when a generalization of classical fuzzy number in the form of an ordered fuzzy number follows? Are the same defuzzification procedures applicable? The answer is partial positive: if the ordered fuzzy number is *proper* one, i.e. its membership relation is a function, then the same procedure can be applied. What to do, however, when the number is improper, i.e. the membership relation is by no means of functional type?

In the case of fuzzy rules in which ordered fuzzy numbers appear as their consequent part we need to introduce a new defuzzification procedure. In this case the concept of functional, even linear, which maps elements of the Banach space into reals, will be useful.

The Banach space $\mathcal{R}$ with its Tichonov product topology of $C([0, 1]) \times C([0, 1])$, with $C([0, 1])$ the Banach space of continuous functions on $[0, 1]$, may lead to a general representation of linear and continuous functional on $\mathcal{R}$. According to the Banach-Kakutami-Riesz representation theorem any linear and continuous functional $\bar{\phi}$ on the Banach space $C([0, 1])$ is uniquely determined by a Radon measure ν on S such that

$$\bar{\phi}(f) = \int_{[0,1]} f(s)\nu(ds) \quad \text{where } f \in C([0, 1]) \ . \tag{6}$$

It is useful to remind that a Radon measure is a regular signed Borel measure (or differently: a difference of two positive Borel measures). A Boreal measure is a measure defined on a σ-additive family of subsets of $[0, 1]$ which contains all open subsets.

However, on the interval $[0, 1]$ each Radon measure is represented by a Stieltjes integral [29] with respect to a function of a bounded variation. Hence we can say that for any continuous functional $\bar{\phi}$ on $C([0, 1])$ there is a function of bounded variation h_ϕ such that

$$\bar{\phi}(f) = \int_0^1 f(s)dh_\phi(s) \quad \text{where } f \in C([0, 1]) \ . \tag{7}$$

Hence we may say that due to the representations (6) and (7) *any linear and bounded functional ϕ on the space $\mathcal{R}$ can be identified with a pair of functions of bounded variation through the following relationship*

$$\phi(f, g) = \int_0^1 f(s)dh_1(s) + \int_0^1 g(s)dh_2(s) \tag{8}$$

where the pair of continuous functions $(f, g) \in \mathcal{R}$ represents an ordered fuzzy number and h_1, h_2 are two functions of bounded variation on $[0, 1]$.

From the above formula an infinite number of defuzzification procedures can be defined. The standard defuzzification procedure in terms of the area under the membership relation can be defined; it is realized by a linear combinations of two Lebesgue measures of $[0, 1]$. In the present case, however, the area is calculated in the y-variable, since the ordered fuzzy number is represented by a pair of continuous functions in the y variable (cf. (2)). Moreover, to each point $s \in [0, 1]$ a Dirac delta (an atom) measure can be related, and such a measure represents a linear and bounded functional which realizes the corresponding defuzzification procedure. For such a functional, a sum (or in a more general case – a linear combination $af(s)+bg(s)$) of their values is attached to a pair of functions (f, g) at this point.

For example, if we take the Dirac atomic measure concentrated at $s = 1$, and define

$$\nu_1 = a\delta_1 \text{ i } \nu_2 = b\delta_1$$

where δ_1 is the atomic measure of $\{1\}$, then the value of the defuzzification operator (functional) in (8), denoted here by ϕ_m and calculated at $A = (f_A, g_A)$ will be

$$\phi_m(A) = af_A(1) + bg_A(1) \tag{9}$$

and if $a + b = 1/2$, then it is a mean value of both functions (from the core of f_A and g_A).

A different choice of the measures may lead to the surface area under the graph of the function, and the first moment of inertia. For example, if

$$\nu_1 = a(s)\lambda \text{ and } \nu_2 = b(s)\lambda$$

where λ is the Lebesgue measure of the interval $[0, 1]$ of the real line, and $a(s), b(s)$ are integrable functions on the interval, then in the case of a positive oriented number $A = (f_A, g_A)$ with $f_A \leq g_A$ and

$$b(s) = -a(s) = 1 \tag{10}$$

the defuzzification functional (8) calculated at $A = (f_A, g_A)$ will give the surface area contained between graphs of f_A and g_A. If, however, in (10) we put

$$b(s) = -a(s) = s \tag{11}$$

we will get the first moment of inertia of this area.

IV. Nonlinear Defuzzification Functionals

It is evident that nonlinear and multivariant function compositions of linear functionals will lead to nonlinear defuzzification functionals. For example, a ratio being a nonlinear composition of two linear functionals, where the first one is the first moment and the second . the surface area, discussed in the previous subsection, may lead to the center of gravity known from the Mamdani approach, however, with respect to $s = y$ variable.

Now we can state a uniform approximation theorem concerning the defuzzification operators (functionals). To this end let us use the following denotation. Let $\mathcal{A} \subset \mathcal{R}$ be a subset of all ordered fuzzy numbers $\mathcal{R}$ formed of pairs of functions which are equi-continuous and equi-bounded. Notice that from the theorem of Ascoli-Arzelà [36] it follows that a subset of $C([0, 1])$ is compact if its elements are equi-continuous and equi-bounded. By $\mathcal{G}$ we denote the set of all multivariant continuous functions defined on the appropriate Cartesian product of the set of real numbers. In other words $F \in \mathcal{G}$ if there is a natural number k such that $F : \mathbf{R}^k \to \mathbf{R}$ and F is continuous in the natural norm of $\mathbf{R}^k$. By $\mathcal{D}$ we denote the set of all linear and continuous functionals defined on $\mathcal{A} \subset \mathcal{R}$ (compare (8)). Here we could identify the set $\mathcal{D}$ with the adjoint space $\mathcal{R}^*$ since each continuous (bounded) and linear functional on the whole space $\mathcal{R}$ is a also continuous, linear functional on each subspace, hence on the subset $\mathcal{A}$. Moreover, each continuous, linear functional on a subspace $\mathcal{A} \subset \mathcal{R}$ can be extended to the whole space $\mathcal{R}$, thanks to the Hahn-Banach theorem [36]. Let us use the denotation $\mathcal{R}^\diamond$ for the space of all continuous (not necessarily linear) functionals from $\mathcal{R}$ into reals $\mathbf{R}$. Notice that $\mathcal{R}^\diamond \supset \mathcal{R}^*$.

If a function F of k variables is from $\mathcal{G}$ and $\varphi_1, \varphi_2, ..., \varphi_k \in \mathcal{D}$ then their superposition $F \circ (\varphi_1, \varphi_2, ..., \varphi_k)$ is a function from $\mathcal{D}$ into $\mathbf{R}$, i.e., the functional

$$F \circ (\varphi_1, \varphi_2, ..., \varphi_k) : \mathcal{D} \to \mathbf{R}, \text{ with } F \in \mathcal{G},\ \varphi_1, \varphi_2, ..., \varphi_k \in \mathcal{D} \tag{12}$$

is a defuzzification operator, nonlinear in general. To make the notation short we will write

$$F \circ (\varphi_1, \varphi_2, ..., \varphi_k) =: F(\varphi_1, \varphi_2, ..., \varphi_k). \tag{13}$$

Theorem. *Let $\mathcal{A} \subset \mathcal{R}$ be a compact subset of the space of all ordered fuzzy numbers $\mathcal{R}$, and let $\mathcal{D}$ be the set of all linear and continuous functionals defined on $\mathcal{A}$, and let $\mathcal{G}$ be the set of all multivariant continuous functions defined on the appropriate Cartesian product of the set of real numbers. Then the set $\mathcal{H}$ composed of all possible compositions (superpositions) of the type (12) where F is from $\mathcal{G}$ and $\varphi_1, \varphi_2, ..., \varphi_k$ are from $\mathcal{D}$, with arbitrary k, is dense in the space $\mathcal{R}^\diamond$ of all continuous functionals from $\mathcal{R}$ into reals **R**.*

The proof based on the classical Stone–Weierstrass theorem will be published in another paper.

V. Center of Gravity

Let us stay with the case of nonlinear functionals. It will be a functional corresponding to that know for convex fuzzy numbers and representing the center of gravity of the area under the graph of the membership function. In the case of OFN the membership function does not exist, in general, however, we may follow to some extend the previous construction.

Let consider an ordered fuzzy number $A = (f, g)$ given in Fig.4. Since $f(g) > g(s)$ for any $s \in [0, 1]$ then by adding an interval (perpendicular to the s-axis) which joints the points $(1, f(1))$ and $(1, g(1))$ we get a figure (an area) bounded by the graphs of $f(s)$ and $g(s)$, and the t–axis. Our aim is to determine the t–th coordinate of the center of gravity of this figure.

Assuming, as it is natural, that the density of each point of the figure is the same and equal to one, first we calculate the moment of inertia of this figure with respect to the t–axis. Here t denotes x variable. Let us consider a differential (incremental) element ds situated between the coordinate values s_1 and s_2 and the corresponding piece of the figure above. Its moment M_s is the product of the area and the length of the arm with respect the t–axis, which is the local center of gravity. The (incremental) area is equal to $[f(s_1) - g(s_1)]ds$, while the center of gravity of this area (its t–coordinate) can be approximated by the middle point of the interval of $[g(s_1), f(s_1)]$ as $\dfrac{f(s_1) - g(s_1)}{2} + g(s_1) = \dfrac{f(s_1) + g(s_1)}{2}$.

Hence we have for the moment of inertia of this differential (incremental) area element the expression

$$\frac{f(s_1) + g(s_1)}{2} [f(s_1) - g(s_1)] ds \,.$$

Since the point s_1 has been chosen quite arbitrarily, the moment of inertia of the whole figure bounded by the graphs of the functions f and g, t–axis and the interval bounding the points $(1, f(1))$ and $(1, g(1))$, will be the integral

$$M = \int_0^1 \frac{f(s) + g(s)}{2} [f(s) - g(s)] ds \tag{14}$$

Let P be the mass of the figure equal to the area of the whole figure, due to our assumption about the homogeneous distribution of the mass,

$$P = \int_0^1 [f(s) - g(s)]ds\,. \tag{15}$$

Now we use the classical balance equation of inertial momentum which states that the moment of inertia M is equal to the product of the mass of the figure P and the arm r (of the figure, which is the t coordinate of the global center of gravity),

$$M = P \cdot r\,. \tag{16}$$

The coordinate r is wanted value of the defuzzification functional representing the center of gravity. Having the expressions (14) and (15) we end up with the following expression for the center of gravity defuzzification functional ϕ_G calculated at OFN (f, g)

$$\phi_G(f,g) = \int_0^1 \frac{f(s)+g(s)}{2}[f(s)-g(s)]ds\{\int_0^1 [f(s)-g(s)]ds\}^{-1}\,. \tag{17}$$

We can see that the functional ϕ_G, denoted on the figures below by COG, is nonlinear. Our derivation is based on the Eq. (16) and the assumption that the function $f(s) \geq g(s)$ which may not be fulfilled in any case. However, if the opposite inequality holds the value does not change. The case when the none of them is true is more complex, we are going, however, to adapt this representation for any ordered fuzzy number (f, g).

VI. Conclusions

The ordered fuzzy numbers are tool for describing and processing vague information. They expand existing ideas. Their "good"ïż¡ algebra opens new areas for calculations. Beside that, new property – *orientation* can open new areas for using fuzzy numbers. Important fact (in authors' opinion) is that thanks to OFNs we can supply without complication the classical field of fuzzy numbers with new ideas. We can use the OFNs instead of convex fuzzy numbers, and if we need to use extended properties we can use them easily. One of directions of the future work with the OFNs is the construction of new class of nonlinear defuzzification functionals based on required properties. The above problem may have several solutions; however, one can look for one of them with the help of Theorem . Since the Weierstrass theorem states that each continuous function (of many variables) defined on a compact set can be approximated with a given accuracy by a polynomial (of many variables) of an appropriate, i.e., sufficiently high order, then with the use of our final result the family $\mathcal{H}$ may be taken as a set of polynomials of many variables. In the recent papers [18], [20] propositions concerning specially dedicated evolutionary algorithms for the determination of the approximate form of the functional have been discussed. Some interpretations and applications of the present approach to

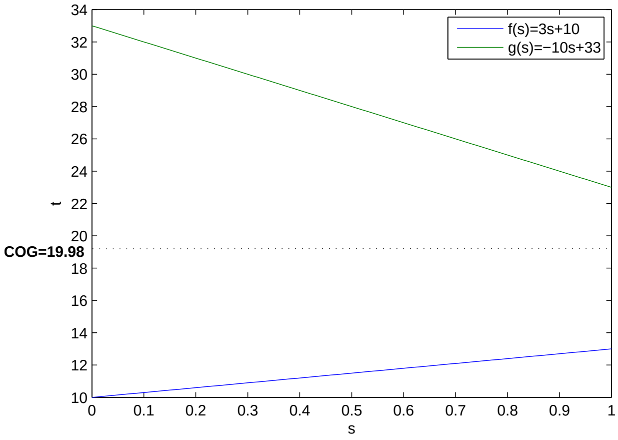

Fig. 2. Ordered fuzzy numbers with affine functions f, g and its center of gravity.

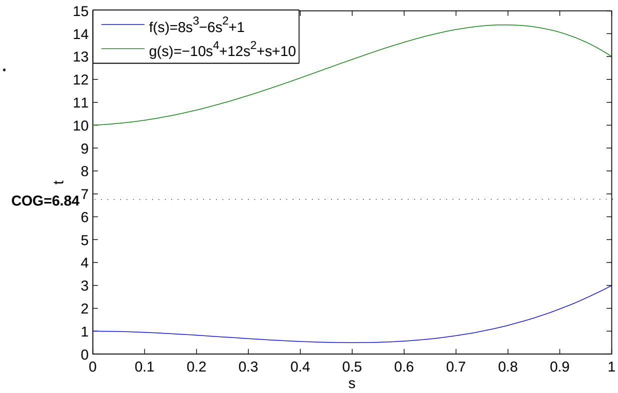

Fig. 3. Ordered fuzzy numbers with polynomial functions f, g and its center of gravity.

fuzzy modeling, control and finance have been presented in the recent publications [17],[19].

Numerical results of implementations of the derived formula for the center of gravity functional $\phi_G =$:COG from Eq. (17) for two ordered fuzzy numbers, with the functions f and g of the affine type (Fig. 2) and of the polynomial type (Fig. 3), respectively, are presented below. On the figures the variable t corresponds to x, and the variable s to y, in Fig.1.

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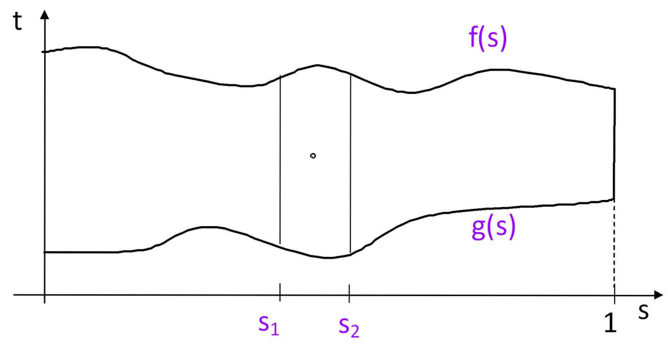

Fig. 4. Sketch of ordered fuzzy number (f, g) and the corresponding area.

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