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Metric properties of populations in artificial immune systems using Hadamard representation

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***Abstract*—A Hadamard representation, which is an alternative towards the binary representation, is considered in this study. It operates on numbers $+1$ and -1. Several properties of such defined representation were pointed out and properties of the immune system were expressed based on this representation.**

***Index Terms*—Genetic algorithm, binary coding, Hadamard representation, artificial immune system.**

I. Introduction

DESPITE the continuous development since the 1960s, the discipline of genetic algorithms [5], [11], [1] is still focused more on the empirical aspects of algorithms than theoretical studies. Methods, which are currently in use in theoretical studies of these algorithms, could be classified into one of the following groups: schema theory [7], markov chains theory [10], dimensional analysis [12], order statistics [4], quantitative genetics [8], orthogonal functions analysis [3], quadratical dynamical systems QDS [14], statistical physics [2].

Despite the fact that sophisticated and complex mathematical models were implemented, neither results were obtained which could have broader field of application (e.g. the schema theory or the Markov chains theory) nor were these methods subject for a vivid discussion. Simplistic assumptions adopted frequently in the theoretical analyses deform the analyzed algorithms in such a way that they question the real connection between the obtained results and the investigated algorithms.

This study undertakes rather unfertile topic of representation of binary chromosomes. In place of the classic, zero-one binary representation, other representation {-1, 1} has been proposed for which the metrics in the binary chromosomes space was determined. This representation, called later the Hadamard representation, allows for an effective proof of the dependence between the whole groups of chromosomes. In the Hadamard representation all chromosomes are equal to the length in a defined metric, and there is no chromosome "'zero"', which often had to be separately discussed in the binary representation due to its distinct properties. Note that in the zero-one binary representation each chromosome can be normalized, and each chromosome has a clearly defined direction except chromosome "'zero"'.

The Hadamard representation was applied for defining properties of the artificial immune systems.

II. Hadamard representation

In 1893, French mathematician Jacques Hadamard in his study *Résolution d'une question relative aux déterminants* [6] presented the properties of a matrix, whose only elements were +1 or -1. In this study, we used this representation as a substitute of a binary representation, omitting the requirement of orthogonal columns pairs. Subject of our deliberation was the following series:

TABLE I
Indexing and representation of points in A_n space.

Element's symbol	Decimal representation	Binary representation	Hadamard representation
h_1	0	(0,0,...,0,0,0)	(1, 1,..., 1, 1, 1)
h_2	1	(0,0,...,0,0,1)	(1, 1,..., 1, 1,-1)
h_3	2	(0,0,...,0,1,0)	(1, 1,..., 1,-1, 1)
⋮	⋮	⋮	⋮
h_{2^n-2}	2^n-3	(1,1,...,1, 0, 1)	(-1,-1,...,-1, 1,-1)
h_{2^n-1}	2^n-2	(1,1,...,1, 1, 0)	(-1,-1,...,-1,-1, 1)
h_{2^n}	2^n-1	(1,1,...,1, 1, 1)	(-1,-1,...,-1,-1,-1)

$$A_n = \{(h_n, h_{n-1}, \ldots, h_2, h_1) : \forall i \in \{1, 2, \ldots, n\} \quad h_i \in \{-1, 1\}\} \quad (1)$$

Its elements represent all possible binary chromosomes of equal length n. We will be considering in our work that n is a natural number higher than 1. The proposed representation has one, apparently insignificant property, which distinguishes it from the binary representation: a square of each coordinates is equaled 1. This fact draws two subsequent conclusions: the sum of the squares of coordinates of each element of the A_n space is constant and equals this space dimension, and there is no element with zero coordinates. The collection of these simple facts allows for the formulation of rules for phenotypes (indices) and development of automate methods of moving frame A_n, as well as determination of the distance (level of differentiation) between the elements of this space.

At the beginning, we determined the order of the indexing of points in A_n and their four representations, which we will use alternating (see Table I).

The number of points included in A_n

$$|A_n| = 2^n \quad (2)$$

For each element in the binary representation there are numerous functions transforming the elements of this representation to the elements of the Hadamard representation and inversely. The same can be said about the Hadamard representation in relation to the binary representation. Three pairs of such functions are presented in Table II.

The distance of points in A_n

The distance of two points $a = (a_n, \ldots, a_2, a_1)$ and $b =$

TABLE II
EXEMPLARY FUNCTIONS TRANSFORMING THE HADAMARD REPRESENTATION INTO THE BINARY REPRESENTATION AND INVERSELY.

For binary representation	For Hadamard representation
$B_1(h_{j,i}) = \begin{cases} 0 & for \quad h_{j,i}=1 \\ 1 & for \quad h_{j,i}=-1 \end{cases}$	$H_1(b_{j,i}) = \begin{cases} 1 & for \quad b_{j,i}=0 \\ -1 & for \quad b_{j,i}=1 \end{cases}$
$B_2(h_{j,i}) = \frac{1-h_{j,i}}{2}$ where $h_{j,i} \in \{-1,1\}$	$H_2(b_{j,i}) = 1-2b_{j,i}$ where $b_{j,i} \in \{0,1\}$
$B_3(h_{j,i}) = \log_{-1} h_{j,i}$ where $h_{j,i} \in \{-1,1\}$	$H_3(b_{j,i}) = (-1)^{b_{j,i}}$ where $b_{j,i} \in \{0,1\}$

$(b_n, \ldots, b_2, b_1)$ in A_n space is measured according to the following equation:

$$\forall a,b \in A_n \qquad w(a,b) = n - \frac{1}{4}\sum_{i=1}^{n}(a_i+b_i)^2 \tag{3}$$

The distance defined in that way will always be a positive integer, which will tell us on what coordinates in Hadamard representation the values differ (exactly as in the binary representation) (see Table I). In addition, A_n space with the w distance determined in this way is metric.

Proof:

1. (identity of indiscernibles)

$$w(a,a) = n-\frac{1}{4}\sum_{i=1}^{n}(a_i+a_i)^2 = n-\frac{1}{4}\sum_{i=1}^{n}4(a_i)^2 = n-n = 0$$

and at the same time
$w(a,b)=0 \Rightarrow n = \frac{1}{4}\sum_{i=1}^{n}(a_i+b_i)^2$. Therefore, for $a_i, b_i \in \{-1,1\}$, $\forall i \in \{1,2,...,n\} \quad a_i = b_i$, we finally arrive at the equation $a=b$.

2. (symmetry)

$$w(a,b) = n-\frac{1}{4}\sum_{i=1}^{n}(a_i+b_i)^2 = n-\frac{1}{4}\sum_{i=1}^{n}(b_i+a_i)^2 = w(b,a).$$

3. (triangle inequality)

$$w(a,b)+w(b,c) =$$

$$= n-\frac{1}{4}\sum_{i=1}^{n}(a_i+b_i)^2 + n-\frac{1}{4}\sum_{i=1}^{n}(b_i+c_i)^2 =$$

$$= 2n-\frac{1}{4}(\sum_{i=1}^{n}(a_i+b_i)^2 + \sum_{i=1}^{n}(b_i+c_i)^2) =$$

$$= 2n-\frac{1}{4}\sum_{i=1}^{n}\left(a_i^2+2b_i\left(a_i+b_i+c_i\right)+c_i^2\right)$$

Due to $a_i, b_i, c_i \in \{-1,1\}$, we obtain the weak inequality $b_i\left(a_i+b_i+c_i\right) \leq a_ic_i+2$. Therefore,

$$\geq 2n-\frac{1}{4}\sum_{i=1}^{n}\left(a_i^2+2\left(a_ic_i+2\right)+c_i^2\right) =$$

$$= 2n-\frac{1}{4}\sum_{i=1}^{n}\left(a_i^2+2a_ic_i+c_i^2\right)-n =$$

$$= n-\frac{1}{4}\sum_{i=1}^{n}\left(a_i^2+2a_ic_i+c_i^2\right) = w(a,c).$$

Hence, we have $w(a,b)+w(b,c) \geq w(a,c)$.

■

Remark R1

For any point established in A_n space, there are $\binom{n}{n-z} = \binom{n}{z}$ different points at exactly z distance.

A set of points in A_n with w metrics constitute the limited metric space of diameter n, it can be easily observed that for any two points $h_t, h_k \in A_n$ there is an inequality

$$w(h_t,h_k) \leq n, \tag{4}$$

and the equality occurs only for $k = 2^n - t + 1$, which is presented in Theorem **T1**.

Lemma L1

$$\forall h_t,h_k \in A_n \qquad w(h_t,h_k)=n \Leftrightarrow$$
$$\forall i \in \{1,2,...,n\} \qquad h_{t,i} = -h_{k,i}$$

Proof:

$$w(h_t,h_k)=n$$
$$\Updownarrow$$
$$n-\frac{1}{4}\sum_{i=1}^{n}(h_{t,i}+h_{k,i})^2 = n$$
$$\Updownarrow$$
$$\sum_{i=1}^{n}(h_{t,i}+h_{k,i})^2 = 0$$

The last equation is true if and only if all the elements of the sum are equal to 0. A single element of the sum $(h_{t,i}+h_{k,i})^2$ is equal to 0 if and only if $h_{t,i} = -h_{k,i}$. ■

Theorem T1

$$\forall t \in \{1,...,n\} \qquad w(h_t,h_k)=n \Leftrightarrow$$
$$ID(h_k) = k = 2^n - t + 1 = 2^n - ID(h_t) + 1$$

where $ID(h_i)$ returns index of h_i.

Proof:

$$\Rightarrow$$

From the Lemma L1, we have $h_{t,i} = -h_{k,i}$. As $h_{j,i} \in \{-1, 1\}$, we conclude that at i-th position one of the element has 1, and the second one -1. Transforming h_t and h_k to a binary representation, we obtain two numbers having the feature, that at i-th position one of the number has 0, and the second one 1. The value of the sum of such elements is equal to $2^n - 1$. At the same time, in a decimal representation h_t and h_k represent $t - 1$ and $k - 1$, respectively. Therefore,

$$(t-1) + (k-1) = 2^n - 1,$$

which is equivalent to

$$k = 2^n - t + 1.$$

$$\Leftarrow$$

Note that the sum of h_t and h_k in a decimal representation is given by

$$(t-1) + (k-1) = (t-1) + (2^n - t) = 2^n - 1$$

Hence, the sum in a binary representation has n-digits 1. Therefore, the components of this sum are the two binary numbers with the property, that at each i-th position one of the components has 1, and the second one 0. Transforming the components into Hadamard representation, we obtain two numbers with the property, that at each i-th position one of the components has -1, and the second one -1. Therefore,

$$h_{t,i} + h_{k,i} = 0$$

$$\frac{1}{4}\sum_{i=1}^{n}(h_{t,i} + h_{k,i})^2 = 0$$

$$w(h_t, h_k) = n$$

■

The points which comply with the above theorem will be called *polar points* and designated as a and $\overline{a}$ pairs. Thus, following equations for the polar points are received:

$$\forall a \in A_n \quad w(a, \overline{a}) = n \tag{5}$$

$$\forall a \in A_n \quad \overline{\overline{a}} = a \tag{6}$$

We can assume:

$$\overline{a} = (\overline{a_n}, \ldots, \overline{a_1}) \tag{7}$$

Lemma **L1** can be presented as follows:

Lemma L2

$$\forall h_k \in A_n \quad \forall i \in \{1, 2, ..., n\} \quad h_{k,i} = -\overline{h_{k,i}}$$

Based on the **R1** remark and **T1** Theorem, it could be concluded that for each point h_t in A_n space, exactly one point h_k occurs in this space, different from h_t, which gives a pair of polar points. Polar points have the following extra two properties

$$\forall c \in A_n \quad \forall a \in A_n \quad w(a, c) + w(c, \overline{a}) = n$$

$$\forall a, b \in A_n \quad w(a, b) = w(\overline{a}, \overline{b})$$

Proof:

$$\forall c \in A_n \quad \forall a \in A_n \quad w(a, c) + w(c, \overline{a}) = n$$

Note that

$$w(a, c) + w(c, \overline{a}) = n$$

$$\Updownarrow$$

$$n - w(a, c) + n - w(c, \overline{a}) = n$$

$$n - w(a, c) + n - w(c, \overline{a}) = \frac{1}{4}\sum_{i=1}^{n}(a_i + c_i)^2 + \frac{1}{4}\sum_{i=1}^{n}(c_i + \overline{a_i})^2 =$$

$$= \frac{1}{4}\sum_{i=1}^{n}(a_i^2 + 2(a_i + c_i + \overline{a}_i)c_i + (\overline{a}_i)^2) =$$

from the Lemma L2, for each $i \in \{1, \ldots, n\}$ we have $a_i + \overline{a_i} = 0$, and $a_i^2 = (\overline{a_i})^2 = c_i^2 = 1$. Hence,

$$= \frac{1}{4}\sum_{i=1}^{n} 4 = n.$$

■

Proof:

$$\forall a, b \in A_n \quad w(a, b) = w(\overline{a}, \overline{b})$$

From Theorem T2 we have

$$w(a, b) + (b, \overline{a}) = n$$

$$w(b, \overline{a}) + w(\overline{a}, \overline{b}) = n$$

From the equality of the right sides, follows the equality of the left sides:

$$w(a, b) + w(b, \overline{a}) = w(b, \overline{a}) + w(\overline{a}, \overline{b})$$

Hence,

$$w(a, b) = w(\overline{a}, \overline{b}).$$

■

After translation to the indices of points of the A_n space, the equation demonstrates another symmetry, besides the one, which results from the second condition of the metric space:

$$w(h_k, h_t) = w(h_{2^n - t + 1}, h_{2^n - k + 1}) \tag{8}$$

The distance $w(h_k, h_t)$ between two points, h_k i h_t, can be estimated according to the **Pod_Od** algorithm. We reduce indices k and t by 1 and divide n times by 2, saving the remainders. The number of differences between the remainders will constitute the sought distance.

Pod_Od Algorithm

Input data:

n - number of positions in representation of point from A_n

k, t - indices of points h_k and h_t

```
begin
  od:=0;
  k:=k-1; t:=t-1;
  i:=0;
```

```
    while i < n do
    begin
      od:=od+((k mod 2)-(t mod 2))^2;
      i:=i+1;
      k:=k div 2;
      t:=t div 2;
    end;
    return od;
  end.
```

During estimation of the distance between two points belonging to the A_n space, the following theorem can be useful.

Theorem T2

$\forall s \in \{0, 1, \ldots, n-1\}$ and $\forall k, t \in \{1, 2, \ldots, 2^s\}$ with $h_k, h_t \in A_n$

$$w(h_k, h_{t+2^s}) = w(h_k, h_t) + 1 \quad (9)$$

$$w(h_{k+2^s}, h_{t+2^s}) = w(h_k, h_t) \quad (10)$$

$$w(h_{k+2^s}, h_{t+2^s}) = w(h_{t+2^s}, h_k) - 1 \quad (11)$$

Proof:

For each $s \in \{0, 1, ..., n-1\}$, if $j = ID(h_j)$ satisfies the inequality $2^{s-1} < j \leq 2^s$ (or equivalent $2^s < j + 2^s \leq 2^{s+1}$), the elements h_j and h_z, where $z = j + 2^s$, differs only at the position $s+1$. The element h_j at this position has the value 1 ($h_{j,s+1} = 1$), and the element h_z -1 ($h_{z,s+1} = -1$).

Therefore, two elements h_t and h_z, where $z = t + 2^s$ while meeting the assumptions of Theorem T2 about s and t, differ at exactly one position only. Hence, if h_k is remote from the element h_t with $w(h_k, h_t)$, then the distance from h_k to the element h_z will be greater by one (according to the Equation (3)), what proves Equation (9). In case of Equation (10), a difference in the value of the coordinate s occurs at the same time in both pairs, i.e., with the k indexes, as well as with t indexes. The simultaneous change in a value on the same coordinate, according to the Equation (3), does not change the value $w(h_k, h_t)$, what proves Equation (10). Equation (11) is to be obtained by using the Equations (10) and (9) and the symmetry rule

$$w(h_{k+2^s}, h_{t+2^s}) = w(h_k, h_t) =$$

$$= w(h_k, h_{t+2^s}) - 1 = w(h_{t+2^s}, h_k) - 1$$

■

Theorem **T2** allows for the construction of an algorithm, which produces the table of distances between any elements from the A_n space. In the **Tab_Od** algorithm, a table was built mechanically (without the estimation of the distances between particular elements). The number situated on the cross of the k-row with the t-column corresponds to the $w(h_k, h_t)$ distance. Therefore, a table of dimension $2n$ could be obtained from the table of dimension n according to the following symbolic equation:

$$\langle W_n \rangle \overset{\mapsto}{\longrightarrow} \left\langle \begin{array}{cc} W_n & \langle W_n + 1 \rangle \\ \langle W_n + 1 \rangle^T & W_n \end{array} \right\rangle$$

where +1 means addition to each element of the table a value of 1.

Tab_Od Algorithm

```
Input data: Enter_size_of_gene  m;
  begin
    n:=1;
    T[n,n]:=0;
    while n<2^m do
    begin
      n:=2*n; {enlarging the table twice}
      {extending all rows, copying the already existing one
      and adding 1 to each expression}
      for i:=1 to (n div 2) do
        for j:=(n div 2)+1 to n do
          T[i,j]:=T[i,j-(n div 2)]+1;

      {copying the new part symmetrically, below the first part}
      for i:= n/2+1 to n do
        for j:=1 to n/2 do T[i,j]:=T[j,i];

      {Pasting a copy of the original square under the calculated
      in the beginning of the loop, as the last part of a new,
      twice as large square}
      for i:= n/2+1 to n do
        for j:=(n div 2)+1 to n do T[i,j]:=T[i-(n div 2),j-(n div 2)];
    end;
  end.
```

The conditions for the distance alignment for $n = 4$ are presented in Table III. The elements were placed according to the order of indices increase. It is worth to mention that the sum of k and $2^n - (k-1)$ indices is constant and equals $2^n + 1$ (with accordance to the **T1** Theorem). Thus, elements calculated in this way constitute a pair of polar points (the distance between polar points in our example is $n = 4$, which lies along the diagonal starting in the right upper corner).

III. Definitions Describing States of Artificial Immune Systems

Artificial immune systems constitute currently a significant trend in the studies on biologically inspired calculations [13]. Idealized states of the artificial immune system are determined in the study and subsequently defined using Hadamard representation. Properties determined in such a way are illustrated by the examples based on the content of Table III.

Radius of tolerance R

A radius of tolerance is understood as the border value enabling a mutual recognition of elements in A_n space.

Two elements $x, y \in A_n$ recognize or do not tolerate each other if the distance between them is higher than the radius of tolerance.

$$w(x, y) > R \quad (12)$$

where R complies with the inequality: $0 \leq R \leq n$.

Elements $x, y \in A_n$ complying the weak inequality

$$w(x, y) \leq R \quad (13)$$

will be described as not recognizing or tolerating each other.

Example 0

In the examples considered here we use the A_n space, whose

TABLE III
TABLE OF DISTANCES BETWEEN ANY ELEMENTS IN A_n SPACE $(n = 4)$.

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**
1	0	1	1	2	1	2	2	3	1	2	2	3	2	3	3	4
2	1	0	2	1	2	1	3	2	2	1	3	2	3	2	4	3
3	1	2	0	1	2	3	1	2	2	3	1	2	3	4	2	3
4	2	1	1	0	3	2	2	1	3	2	2	1	4	3	3	2
5	1	2	2	3	0	1	1	2	2	3	3	4	1	2	2	3
6	2	1	3	2	1	0	2	1	3	2	4	3	2	1	3	2
7	2	3	1	2	1	2	0	1	3	4	2	3	2	3	1	2
8	3	2	2	1	2	1	1	0	4	3	3	2	3	2	2	1
9	1	2	2	3	2	3	3	4	0	1	1	2	1	2	2	3
10	2	1	3	2	3	2	4	3	1	0	2	1	2	1	3	2
11	2	3	1	2	3	4	2	3	1	2	0	1	2	3	1	2
12	3	2	2	1	4	3	3	2	2	1	1	0	3	2	2	1
13	2	3	3	4	1	2	2	3	1	2	2	3	0	1	1	2
14	3	2	4	3	2	1	3	2	2	1	3	2	1	0	2	1
15	3	4	2	3	2	3	1	2	2	3	1	2	1	2	0	1
16	4	3	3	2	3	2	2	1	3	2	2	1	2	1	1	0

distance tables are presented in Table III. Moreover, for all the demonstrated examples we assume the value of the radius of tolerance $R = 2$.

Self-aggression
System $B_k \subseteq A_n$ undergoes self-aggression if elements x, y occur, which recognize each other and belong to this system.

$$\exists x, y \in B_k \ : \quad w(x, y) > R$$

Example 1
In A_4 , the systems undergoing self-aggression are for example:
$B_8 = \{h_1, h_2, h_3, h_5, h_9, h_4, h_6, h_7\}$ where $w(h_2, h_7) = 3$
$B_4 = \{h_4, h_6, h_7, h_{10}\}$ where $w(h_7, h_{10}) = 4$
System $B_k \subseteq A_n$ is free of self-aggression if any two elements belonging to this system do not recognize themselves.

$$\forall x, y \in B_k \qquad w(x, y) \leq R$$

Example 2
Free systems of self-aggression, when $R = 2$:
$B_5 = \{h_1, h_2, h_3, h_5, h_9\}$
$B_3 = \{h_4, h_6, h_7\}$
$B_2 = \{h_1, h_2\}$
Let us notice that system B_2 is free of self-aggression also when $R = 1$.

Striking distance
A striking distance of a system $B_k \subseteq A_n$ is a series of points of A_n recognized by any point of B_k .

$$P(B_k) = \{z \in A_n : \quad \exists x \in B_k \quad \wedge \quad w(x, z) > R\}$$

Example 3
For $B_3 = \{h_4, h_6, h_7\}$ from the **Example 2** the striking distance is:

$$P(B_3) = \{h_2, h_3, h_5, h_9, h_{11}, h_{12}, h_{13}, h_{14}, h_{15}\}$$

If B_k undergoes self-aggression then some points belonging to B_k simultaneously belong to $P(B_k)$, which means that

$$B_k \cap P(B_k) \neq \emptyset$$

Example 4
In such state occurs system $B_4 = \{h_4, h_6, h_7, h_{10}\}$ from the **Example 1**:

$$P(B_4) = \{h_2, h_3, h_5, h_7, h_8, h_9, h_{10}, h_{11}, h_{12}, h_{13}, h_{14}, h_{15}\}$$

thus:

$$B_4 \cap P(B_4) = \{h_7, h_{10}\} \neq \emptyset$$

Otherwise, if B_k is free of self-aggression, then B_k and $P(B_k)$ are disjunctive series, which can be presented as follows:

$$B_k \cap P(B_k) = \emptyset$$

Example 5
Free system of self-aggression is B_3, described in **Example 2** and **3**, for which identity occurs:

$$B_3 \cap P(B_3) = \emptyset$$

Complete system
System B_k is complete if its striking distance contains its whole completion $\overline{B_k} = A_n \backslash B_k$.

$$\overline{B_k} \subseteq P(B_k)$$

Example 6
The conditions of the complete system are fulfilled by $B_5 = \{h_1, h_2, h_3, h_4, h_5\}$, for which following identities occur:

$$P(B_5) = \{h_4, h_5, h_6, h_7, h_8, h_9, h_{10}, h_{11}, h_{12}, h_{13}, h_{14}, h_{15}\}$$

$$\overline{B_5} = \{h_6, h_7, h_8, h_9, h_{10}, h_{11}, h_{12}, h_{13}, h_{14}, h_{15}\},$$

Thus, a relation occurs:

$$\overline{B_5} \subseteq P(B_5)$$

Balanced system
System B_k is balanced if at the same time it is a system free of self-aggression, and complete.

$$\overline{B_k} = P(B_k)$$

Example 7

This time let us assume that $B_5 = \{h_1, h_2, h_3, h_5, h_9\}$. For such a system the following identities are fulfilled:

$$P(B_5) = \{h_4, h_6, h_7, h_8, h_{10}, h_{11}, h_{12}, h_{13}, h_{14}, h_{15}\}$$

$$\overline{B_5} = \{h_4, h_6, h_7, h_8, h_{10}, h_{11}, h_{12}, h_{13}, h_{14}, h_{15}\}$$

and

$$\overline{B_5} = P(B_5)$$

Extensive system

We call $B_k \subseteq A_n$ an extensive system if any crossing of its elements results in offspring, which also belongs to this system.

$$\forall x, y \in B_k \subseteq A_n \quad \forall c \in \{0, 1, \ldots, n\} \quad K(\{x, y\}, c) \subseteq B_k$$

where $K(\{x, y\}, c)$ denotes the crossing operation of the two elements x, y from the A_n space in the point c, and is defined as follows:

Let $r_t, r_k \in A_n$, where $r_t = (r_{t,n}, r_{t,n-1}, \ldots, r_{t,2}, r_{t,1})$, $r_k = (r_{k,n}, r_{k,n-1}, \ldots, r_{k,2}, r_{k,1})$.

For $0 \leq c \leq n$,

$$K(\{r_t, r_k\}, c) \mapsto \{(r_{t,n}, r_{t,n-1}, \ldots, r_{t,c+1}, r_{k,c}, r_{k,c-1}, \ldots, r_{k,1}), (r_{k,n}, r_{k,n-1}, \ldots, r_{k,c+1}, r_{t,c}, r_{t,c-1}, \ldots, r_{t,1})\} \quad (14)$$

Example 8

The examples of the extensive systems are presented below:

$$B_2 = \{h_1, h_2\}$$

$$B_4 = \{h_1, h_2, h_3, h_4\}$$

To check the extensibility of B_2 and B_4 systems, equations given by [9] can be used. It can be noticed that both the singleton system and any A_n space, as a whole, are extensive systems.

Expansive system

A system is expansive if it possesses elements (not necessary different), which after a peculiar crossing produce elements out of the system.

$$\exists x, y \in B_k \subseteq A_n \quad \exists c \in \{0, 1, \ldots, n\} : \quad K(\{x, y\}, c) \not\subset B_k$$

Example 9

Let us assume $B_3 = \{h_1, h_2, h_3\} \subseteq A_4$. After crossing on chromosomes h_2 and h_3, performing cuts between the first and the second allele, we obtain chromosomes h_1 and h_4. Chromosome $h_4 \notin B_3$ represents an expansive character. In the study [9], equations allowing calculation of crossings' results directly on chromosomes indices, without the need to recreate their internal structure, could be found. The conclusion of those equations is:

$$K(\{h_2, h_3\}, 1) \mapsto \{h_1, h_4\} \not\subset B_3$$

IV. Summary

In this study a so called Hadamard representation was implemented. It allows to prove the dependence between subsequent generations of binary chromosomes. Some properties of this representation were pointed out, which allows for quick and simple operations on chromosomes indices, instead of processing the binary sequences.

Hadamard representation was applied in a brief definition of some properties of the artificial immune systems. This representation will also allow employing natural numbers in calculating the results of such operations like crossing, mutation, or genetic inversion, as well as determining the influence of these operations on the whole concerned population.

Introduced concepts allow a distinction and classification of different populations, which allows us to ponder the future potential directions of their evolution (states reachable, unreachable, etc.) regardless of the crossing algorithm. Further, it should allow us to compare the genetic algorithms in terms of effectiveness and optimization! Comparing the two algorithms, we must ensure comparability of the populations in which we conduct experiments. It is obvious that the same algorithm for example in the population of the class of expansive systems, has a chance of finding new solutions in subsequent generations, but populations with extensive class, after reviewing the current population, better solutions will no longer find. Self-aggression systems have dispersed chromosomes in the space, as opposed to the free systems of self-aggression, which are concentrated in the vicinity of a chromosome. For such systems we can have a suspicion that for the purpose of continuous functions we have to deal with a local extremum. In the case of complete systems, we can be sure that we control the entire space under consideration, although we use only a separate part of the chromosomes of that space. It is important in many cases to set minimum-complete systems for the space in question.

The presentation of the majority of the basic genetic operators known in the literature as Hadamard representation is planned, as well as studies on the analytical demonstration of temporal properties of the genetic operators and the artificial immune system expressed an {+1, -1} notation will be carried out.

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