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Extrapolation of Non-Deterministic Processes Based on Conditional Relations

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***Abstract*—A problem of extrapolation of a large class of processes and of their future states forecasting based on their occurrence in the past is considered. Discrete-time discrete-value processes are presented as instances of relations subjected to the general relations algebra rules. The notions of relative relations, parametric relations and non-deterministic relations have been introduced. For extrapolated process states assessment relative credibility levels of process trajectories are used. The variants of direct one-step, indirect one-step and direct multi-step process extrapolation are described. The method is illustrated by numerical examples.**

I. Introduction

EXTRAPOLATION of processes is a basis of decision making in control, management, administration, social and/or economical planning, governing, etc. Roughly speaking, it consists in predicting future states of a process on the basis of its known current and past states as well as of some knowledge concerning influencing the process circumstances. Even extrapolation of deterministic processes, whose form can a priori be calculated using some analytical rules, may become a non-trivial problem. A process described by nonlinear differential equations may not only need high computational costs for being predicted but in certain cases it cannot be for long periods exactly predicted because of extreme equations' sensitivity to the initial conditions which with limited accuracy only can be established [1]. Extrapolation of non-deterministic processes is charged by inevitable uncertainty. In the simplest case of stochastic processes extrapolation the uncertainty takes the form of a statistical extrapolation error [2]. The rate of the error can be calculated when the probability distribution of the process is a priori known; otherwise, it should be experimentally evaluated [3, 4]. However, when non-probabilistic indeterminism of the processes takes place, the problem of based on the decision making, including extrapolation, becomes much more complicated [5]. On the other hand, namely non-deterministic processes whose future states are to be predicted in many application areas play an important role. The effectiveness of statistical methods of process extrapolation is satisfactory when the analyzed process's behavior depends on numerous unknown factors of approximately comparable strength. In a more general case of non-deterministic processes, some of the factors may be dominating, while their number and individual properties (e.g. nature, parameters, way of influence on the extrapolated process, etc.) remain unknown their existence by final effects only being manifested. Effective predicting of future states of such processes, on one hand, depends on making use of all available and relevant information, and on the other one, on neglecting all side information making the prediction incorrect. Till now, no commonly accepted definition of non-deterministic processes, excepting some of their particular cases, exists. In this paper an attempt to non-deterministic processes extrapolation based on extended theory of relations [6,7] is presented. Roughly speaking, it is assumed that the past, current and future states of an analyzed process as elements of an universe *U* are instances of a fuzzy multivariable relation; the problem of process extrapolation can thus be interpreted as the one of conditional relation identification when its instances are partially known. Moreover, some general concepts of relative logic (more widely presented in [9,10]) to an evaluation of the extrapolated states of the processes are used. The aim of this paper is to show that extrapolation of non-deterministic processes even if no numerical their instances' logical value is available is still possible. Such conclusion seems to be important in the case of extremely high indeterminism level of the considered processes.

The paper is organized as follows: preliminary extended relation theory concepts are described in Sec. II; in Sec III process extrapolation rules based on relations are presented. An example of process extrapolation under uncertainty is also given there. Conclusions and suggestions concerning future works are presented in Sec. IV.

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II. Extended Fuzzy Relations

It will be considered a countable and linearly ordered family F of mutually isomorphic sets $U^{(i)}$, $i \in [\ldots -2, -1, 0, 1, 2, \ldots]$. Within this paper the indices i will be interpreted as discrete: negative – past, null – current, positive – future time-instants. The family F will be called an universe, the sets $U^{(i)}$ – instant state repertories and their elements $\mathbf{u}^{(i)}_k$ – instant states. Any subfamily $F^{(m)} \subseteq F$, m = 1,2,3,…, preserving the linear order of F will be called its sub-universe; the family of all sub-universes of F (including F itself as well as empty subfamily $\varnothing$) will be denoted by Ω_F. For each sub-universe $F^{(m)}$ a Cartesian product of its instant state repertories:

$$C^{(m)} = \times_{(i)} U^{(i)}, \quad U^{(i)} \in F^{(m)}, \quad F^{(m)} \in \Omega_F, \tag{1}$$

will be called a repertory of scenarios. Any subset

$$R^{(m)} \subseteq C^{(m)} \tag{2}$$

is a relation described in the sub-universe $F^{(m)}$ called below a process while its instances $\mathbf{y} \in R^{(m)}$ will be called process trajectories. A process is thus defined as a set of trajectories described in a given universe F or in any its sub-universe $F^{(m)}$. Any subset $R^* \subset R^{(m)}$ is called a partial process (a partial relation) of $R^{(m)}$ while any projection $R^{(p)} = R^{(m)}{}_{C(p)}$ of $R^{(m)}$ on a repertory of scenarios $C^{(p)}$ corresponding to a sub-universe $F^{(p)} \subset F^{(m)}$ is called a sub-process (a sub-relation) of $R^{(m)}$. A partial process consists thus of selected trajectories while a sub-process consists of cancelled (e.g. to a shortened time- interval) sub-trajectories of $R^{(m)}$. In the above-mentioned cases $R^{(m)}$ is called a broadening of R* and an extension of $R^{(p)}$.

For a given sub-universe $F^{(m)}$ we denote by $\Omega^{(m)}$ the family of all processes, including an empty process $\theta^{(m)}$ and a "trivial" process $C^{(n)}$, that on $F^{(m)}$ can be defined. Similarly, it will be denoted by Ω_F the family of all processes that can be defined on any sub-families $F^{(m)}$ of the universe F. Then, an extended algebra of relations that can be described on F and its sub-universes is given by a quintuple [7]:

$$A_F = [F, \Omega_F, \underline{\cup}, \underline{\cap}, \underline{\neg},] \tag{3}$$

where $\underline{\cup}$, $\underline{\cap}$, $\underline{\neg}$, stand, respectively, for extended sum, intersection and negation of relations. Let us remark that an extended difference of relations $R^{(m)} \underline{\setminus} R^{(n)}$ is given by the formula:

$$R^{(m)} \underline{\setminus} R^{(n)} = R^{(m)} \underline{\cap} (\underline{\neg} R^{(n)}). \tag{4}$$

All concepts of the extended algebra of relations can thus directly be applied to the processes.

Example 1

Let I be a countable set of real integers i called time-instants and let U be a countable set of elements u_k, $k \in [\ldots,-2, -1, 0, 1, 2,\ldots]$ called states. Then any function:

$$f_\nu : I \to U \tag{5}$$

$\nu \in N$, where N denotes a set of natural numbers, describes a discrete process trajectory (whose interpretation depends on a given application area). We take into consideration four time-instants: $i_1, i_2, i_3, i_4 \in I$ such that $i_1 < i_2 \le i_3 < i_4$ and two discrete time-intervals: $I', I'' \subseteq I$, $I' = [i_1,\ldots,i_3]$, $I'' = [i_2,\ldots,i_4]$. Evidently, it is $I' \cap I'' \neq \varnothing$ where $\varnothing$ denotes an empty set. We shall denote by $f_\nu|_{I'}$ and $f_\nu|_{I''}$ the function f_ν cancelled, respectively, to the time-interval I' and I''. The above-described notions can also be interpreted in the relations theory terms. For this purpose an universe takes the form of linearly ordered family of sets $F = U^{I}$. Their Cartesian product C_F plays the role of a multidimensional discrete space whose elements (discrete vectors) correspond to the process trajectories. We also shall denote by $F' = U^{I'}$ and $F'' = U^{I''}$ the sub-universes of F consisting of the sets marked by indices belonging, respectively, to the discrete time-intervals I' and I''; the corresponding Cartesian products (repertories of scenarios) will be denoted by C' and C''. Any process trajectory $\mathbf{y} \in C_F$ outlines thus in unique way in C' and C'' process sub-trajectories $\boldsymbol{y'}$ and $\boldsymbol{y''}$ called, respectively, the projections of **y** on the intervals I' and I'' as shown in Fig. 1 (the sub-trajectories are marked in gray, their common part is shown in black color). On the other hand, the process trajectory **y** will be called an extension of $\boldsymbol{y'}$ (respectively, of $\boldsymbol{y''}$) on the interval I.

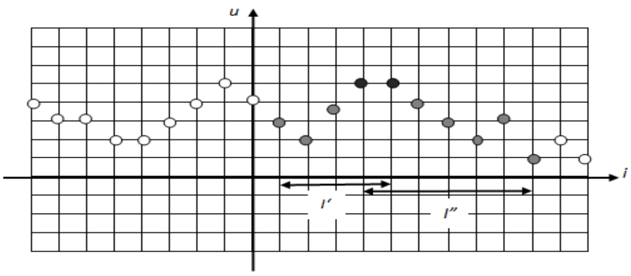

Fig. 1. Example of a discrete process trajectory **y** and its sub-trajectories $\boldsymbol{y'}$ and $\boldsymbol{y''}$.

Any set of discrete process trajectories $\mathbf{Y^*} = \{\mathbf{y}^{(\nu)}\}$, $\nu \in N$, described on a given interval $I^* \subseteq I$, is a subset of Cartesian product $\mathbf{Y^*} \subseteq C^*$; hence, it defines a relation in the family of sets F*. On the other hand, each relation defined in F* describes a discrete process whose trajectories constitute the instances of the relation. As it has been mentioned above, on the basis of the family Ω_F of all relations that can be defined on the sub-families of F it can be defined an extended algebra A_F of relations; it also becomes an algebra of discrete processes described in C_F.

Let us denote by $\boldsymbol{Y'}$ and $\boldsymbol{Y''}$ two discrete processes described, respectively, on the (not obviously disjoint) intervals I' and I''. Then:

a) an extended sum $\boldsymbol{Y'} \,\bar{\underline{\cup}}\, \boldsymbol{Y''}$ of the processes is a discrete process defined on the interval $I' \cup I''$ as a set of all trajectories $\mathbf{y}^{(\nu)}$ such that their projections on I' are the trajectories of ***Y'*** or their projections on *I"* are the trajectories of ***Y"***;

b) an extended product $\boldsymbol{Y'} \sqcap \boldsymbol{Y''}$ of the processes is a

c) discrete process defined on the interval $I' \cup I''$ as a set of all trajectories $\mathbf{y}^{(\nu)}$ such that their projections on *I'* are the trajectories of ***Y'*** and their projections on *I"* are the trajectories of ***Y"***;

d) an extended difference $\boldsymbol{Y'} \setminus \boldsymbol{Y''}$ of the processes is a discrete process defined on the interval $I' \cup I''$ as a set of all trajectories $\mathbf{y}^{(\nu)}$ such that their projections on *I'* are the trajectories of ***Y'*** and their projections on *I"* are not the trajectories of ***Y"***.

The following example will illustrate the above-given notions. Two discrete time-intervals are defined as follows (see Fig. 1):

$$I' = [1,2,3,4,5], \quad I'' = [4,5,6,7,8,9,10];$$

(the intervals are overlapping at the time-instants 4 and 5).

In these intervals two discrete processes are given by their trajectories:

$\boldsymbol{Y'}$ = {[3,2,4,5,5], [2,3,4,5,4], [5,3,4,6,5]},
$\boldsymbol{Y''}$ = {[3,2,4,5,5,6,4], [5,5,4,3,2,3,1], [6,5,0,3,4,5,5], [1,2,0,3,4,6,7]}.

Their algebraic combinations will be described as processes in a joint time-interval:

$$I = [1,2,3,4,5,6,7,8,9,10].$$

Then one obtains the following algebraic combinations of the processes:

$\boldsymbol{Y'} \sqcup \boldsymbol{Y''}$ = {[3,2,4,5,5,*,*,*,*,*], [2,3,4,5,4,*,*,*,*,*], [5,3,4,6,5,*,*,*,*,*], [*,*,*,3,2,4,5,5,6,4], [*,*,*,5,5,4,3,2,3,1], [*,*,*,6,5,0,3,4,5,5], [*,*,*,1,2,0,3,4,6,7];

where * denotes any element (instant state) of U;

$\boldsymbol{Y'} \sqcap \boldsymbol{Y''}$ = {[3,2,4,5,5,4,3,2,3,1], [5,3,4,6,5,0,3,4,5,5]};
$\boldsymbol{Y'} \setminus \boldsymbol{Y''}$ = {[2,3,4,5,4,*,*,*,*,*]},
$\boldsymbol{Y''} \setminus \boldsymbol{Y'}$ = {[*,*,*,3,2,4,5,5,6,4], [*,*,*,1,2,0,3,4,6,7]}.

Negations of relations (complements of relations) can be defined as extended differences of trivial relations and the relations under consideration:

$\neg \boldsymbol{Y'} \equiv C_F \setminus \boldsymbol{Y'}$,
$\neg \boldsymbol{Y''} \equiv C_F \setminus \boldsymbol{Y''}$,

It can be remarked that the extended sum $\boldsymbol{Y'} \sqcup \boldsymbol{Y''}$ represent bunches of trajectories satisfying ***Y'*** in the time-interval *I'* or ***Y"*** in the time-interval *I"* (or satisfying both of them), while the extended intersection $\boldsymbol{Y'} \sqcap \boldsymbol{Y''}$ can geometrically be interpreted as a consistent prolongation of ***Y'*** by ***Y"***. The intersection of relations defines also the following two sub-relations of ***Y'*** and **Y"** called their conditional relations:

$\boldsymbol{Y'}|\boldsymbol{Y''} = (\boldsymbol{Y'} \sqcap \boldsymbol{Y''})_{F'} \times C_{F''\setminus F'} =$
= {[3,2,4,5,5,*,*,*,*,*], [5,3,4,6,5,*,*,*,*,*]},
$\boldsymbol{Y''}|\boldsymbol{Y'} = C_{F'\setminus F''} \times (\boldsymbol{Y'} \sqcap \boldsymbol{Y''})_{F''} =$
={[*,*,*,5,5,4,3,2,3,1], [*,*,*,6,5,0,3,4,5,5]},

(lower indices at the Cartesian products and relations denote their projections on the given sub-families of sets) read, respectively, as: "***Y'*** assuming that ***Y"*** " and "***Y"*** assuming that ***Y'*** ".

A pair of relations ***Y'*** and ***Y"*** is called mutually independent if $\boldsymbol{Y'} \sqcap \boldsymbol{Y''} \equiv \boldsymbol{Y'} \times \boldsymbol{Y''}$; in such case it is $(\boldsymbol{Y'}|\boldsymbol{Y''})_{F'} \equiv \boldsymbol{Y'}$ and $(\boldsymbol{Y''}|\boldsymbol{Y'})_{F''} \equiv \boldsymbol{Y''}$ •

The conditional relations play a substantial role in process extrapolation. From a formal point of view all algebraic operations illustrated in Example 1 can also be interpreted as super-relations (relations between relations): a/ described on a composite Cartesian product $C' \times C'' \times C_{F' \cup F''}$ in the case of extended sum, intersection or differences and b/ on $C' \times C_{F' \cup F''}$ or on $C'' \times C_{F' \cup F''}$ in the case of complementary relations.

Till now, deterministic relations and their application to deterministic processes description were considered. A notion of parametric relation is a step to non-deterministic relations definition. Let $W^{(m)}$ be a set of real parameter values. We take into consideration a Cartesian product $C^{(m)}$ and a relation $R^{(m)}$ as defined by (1) and (2). A Cartesian product

$$D^{(m)} = C^{(m)} \times W^{(m)} \tag{6}$$

and a relation

$$P^{(m)} \subseteq D^{(m)} \tag{7}$$

such that: 1) the projection $P^{(m)}|_{C^{(m)}}$ of $P^{(m)}$ on $C^{(m)}$ is identical to $R^{(m)}$, 2) to each instance of $R^{(m)}$ exactly one value w, $w \in W^{(m)}$, is assigned. Apparently, if additional parameters are considered as other relations' variables, the nature of the relation is not changed and the extended relations algebra rules hold as well. However, this means that following from this algebra rigid rules of inheritance of parameter values on algebraic combinations of the trajectories are imposed. For the sake of practical utility parametric relation should be defined rather so that parameter transformation rules are defined independently on the algebraic operation rules. Parametric relations are then created by parameterization of some ordinary relations consisting in: 1st choosing the sets of parameters W, 2nd assigning parameter values to the instances of the relations, and 3rd establishing a set Q of functional relations:

$$q_{\cup} \subseteq W^{(m)} \times W^{(n)} \times W, \tag{8a}$$
$$q_{\cap} \subseteq W^{(m)} \times W^{(n)} \times W, \tag{8b}$$
$$q_{\setminus} \subseteq W^{(m)} \times W^{(m)} \times W, \tag{8c}$$

assigning in unique way the values of parameters, respectively, to the instances of extended sums, intersections, and differences of relations. The extended algebraic operations on parametric relations can then be defined as super-relations:

$$P^{(m)} \cup^* P^{(n)} = [(P^{(m)} \bar{\cup} P^{(n)}) \bar{\cap} q_{\cup}]_{R(m)\times R(n)\times W}, \quad (9a)$$
$$P^{(m)} \cap^* P^{(n)} = [(P^{(m)} \bar{\cap} P^{(n)}) \bar{\cap} q_{\cap}]_{R(m)\times R(n)\times W}, \quad (9b)$$
$$P^{(m)} \setminus^* P^{(n)} = [(P^{(m)} \bar{\setminus} P^{(n)}) \bar{\cap} q_{\setminus}]_{R(m)\times R(n)\times W}, \quad (9c)$$

which mean that 1st the algebraic operations should separately be performed on the component relations and on the assigned to them parameters, 2nd their results should be integrated by intersection of relations, and 3rd redundant parameter values from the super-relations should be removed by their projection on reduced sub-families of states. The form of the relations $q_{\cup}$, $q_{\cap}$ and $q_{\setminus}$ is not predetermined, it depends on the application purposes. If a family of parametric relations in the same universe F is considered, a family G_W of parameter sets instead of a single set W should be taken into consideration.

For a given universe F, its Cartesian product C_F, a family G_W of sets of parameters, a family Q of parameters transformations and a family Ω_D of all parametric relations reached by parameterization of the relations of Ω_F an algebra of parametric relations can be defined as an 8-tuple:

$$A_P = [F, C_F, G_W, Q, \Omega_D, \cup^*, \cap^*, \setminus^*]. \quad (10)$$

Example 2

There will be considered the relations ***Y'*** and ***Y"*** described in Example 1. It is established a set of parameters W = [0,…,1] and the parameters are assigned to the trajectories as follows:

P' = {[3,2,4,5,5; 0.2)], [2,3,4,5,4; 0.6), [5,3,4,6,5; 0.3)]},

P"={[3,2,4,5,5,6,4; 0.3], [5,5,4,3,2,3,1; 0.1], [6,5,0,3,4,5,5; 0.5], [1,2,0,3,4,6,7; 0.4]} (parameters being separated from other components by semicolons). It is assumed that the parameters will be assigned to the sum and intersection of the relations in the below-described way.

If w_p, $w_q \in W$ are parameter values assigned, respectively, to two given trajectories $\boldsymbol{u'_p} \in \boldsymbol{P'}$, $\boldsymbol{u''_q} \in \boldsymbol{P''}$ then:

1. if $\boldsymbol{u'_p} \bar{\cup} \boldsymbol{u''_q}$ satisfies ***P'*** only then parameter value w_p is to it assigned;
2. if $\boldsymbol{u'_p} \bar{\cup} \boldsymbol{u''_q}$ satisfies ***P"*** only then parameter value w_q is to it assigned;
3. if $\boldsymbol{u'_p} \bar{\cup} \boldsymbol{u''_q}$ satisfies both ***P'*** and ***P"*** then parameter value $\max(w_p,w_q)$ is to it assigned;
4. for $\boldsymbol{u'_p} \bar{\cap} \boldsymbol{u''_q}$ the parameter value $\min(w_p,w_q)$] is to it assigned;

According to this the following parameter values (weights) to the trajectories will be assigned:

P' ∪* ***P"*** = {[**3,2,4,5,5**,4,5,5,6,4; 0.2], [**2,3,4,5,4,**4,5,5,6,4; 0.6], [**5,3,4,6,5**,4,5,5,6.4;0.3], [**3,2,4,5,5,4,3,2,3,1**; 0.2)], [**2,3,4,5,4,**4,3,2,3,1; 0.6], [**5,3,4,6,5**,4,3,2,3.1;0.3], [**3,2,4,5,5**,0,3,4,5,5; 0.2)], [**2,3,4,5,4,**0,3,4,5,5; 0.6], [**3,2,4,5,5**,0,3,4,6,7; 0.2)], [**2,3,4,5,4,**0,3,4,6,7; 0.6], [**5,3,4,6,5**,0,3,4,6.7;0.3], [3,2,4,**3,2,4,5,5,6,4**; 0.3], [2,3,4,**3,2,4,5,5,6,4**; 0.3], [5,3,4,**3,2,4,5,5,6,4**; 0.3], [2,3,4,**5,5,4,3,2,3,1**; 0.1], [5,3,4,**5,5,4,3,2,3,1**; 0.1], [3,2,4,**6,5,0,3,4,5,5**; 0.5], [2,3,4,**6,5,0,3,4,5,5**; 0.5], [**5,3,4,6,5,0,3,4,5,5**; 0.5], [3,2,4,**1,2,0,3,4,6,7**; 0.4], [2,3,4,**1,2,0,3,4,6,7**; 0.4], [5,3,4,**1,2,0,3,4,6,7**; 0.4]};

P' ∩* ***P"*** = {[**3,2,4,5,5,4,3,2,3,1**;0.1], [**5,3,4,6,5,0,3,4,5,5;** 0.3]};

¬****P'*** = (*C'* \ ***F'***) ×{0.4};

¬****P"*** = (*C"* \ ***F"***) ×{0.5},

where the sub-trajectories which contributed to the resulting parameter values have been indicated in bold.

Let us remark that in the above-given case the parameters are not additive; they may reflect relative "importance", "reliability" or other sort of "usefulness" of the trajectories. Moreover, to all trajectories of a negation (¬****P'*** or ¬****P"***) the same parameter values (0.4 or 0.5, respectively) have been assigned •

Example 2 shows a way to blurring (fuzzying) the relations: fuzzy relation is a parametric relation whose parameter values assigned to the instances reflect a level of credibility that the relation is by them satisfied. The expressions (8a-c, 9) hold thus for the fuzzy relations assuming that the sub-relations $q_{\cap}$, $q_{\cup}$, $q_{\neg}$ have been chosen adequately to instances' credibility description purposes. For this purpose they should satisfy some general triangular norms and co-norms conditions [9]. Such conditions are satisfied not only in the above- presented Example 2 but also in some other cases. Among them the following one deserves some attention.

Example 3

Once again, the relations ***Y"*** and ***Y"*** will be taken into consideration. However, instead of exact numerical credibility values their symbolic denotations to their instances will be assigned as follows:

P' = {[3,2,4,5,5; ε_1)], [2,3,4,5,4; ε_5), [5,3,4,6,5; ε_2)]},

P" ={[3,2,4,5,5,6,4; ε_2], [5,5,4,3,2,3,1; ε_0], [6,5,0,3,4,5,5; ε_4], [1,2,0,3,4,6,7; ε_3]}

under additional assumptions:

$$0 < \varepsilon_0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_3 < \varepsilon_4 < \varepsilon_5 < \varepsilon_6 ,$$

level ε_0 being assigned to all instances not belonging to ***P'*** or to ***P"***.

Then, according to the formerly presented rules 1 – 4, the following credibility values to the instances of algebraic combinations of relations will be assigned:

P' ∪* ***P"*** = {[**3,2,4,5,5**,4,5,5,6,4; ε_2], [**2,3,4,5,4**,4,5,5,6,4; ε_6], [**5,3,4,6,5**,4,5,5,6.4; ε_3], [**3,2,4,5,5,4,3,2,3,1**; ε_2)], [**2,3,4,5,4**,4,3,2,3,1; ε_6], [**5,3,4,6,5**,4,3,2,3.1; ε_3],

[**3,2,4,5,5,**0,3,4,5,5; ε_2)],
[**2,3,4,5,4,**0,3,4,5,5; ε_6],
[**3,2,4,5,5,**0,3,4,6,7; ε_2)],
[**2,3,4,5,4,**0,3,4,6,7; ε_6], [**5,3,4,6,5,**0,3,4,6.7; ε_3],
[3,2,4,**3,2,4,5,5,6,4**; ε_3],
[2,3,4,**3,2,4,5,5,6,4**; ε_3], [5,3,4,**3,2,4,5,5,6,4**; ε_3],

[2,3,4,**5,5,4,3,2,3,1**; ε_1], [5,3,4,**5,5,4,3,2,3,1**; ε_1],
[3,2,4,**6,5,0,3,4,5,5**; ε_5],
[2,3,4,**6,5,0,3,4,5,5**; ε_5], [**5,3,4,6,5,0,3,4,5,5**; ε_5],
[3,2,4,**1,2,0,3,4,6,7**; ε_4],
[2,3,4,**1,2,0,3,4,6,7**; ε_4], [5,3,4,**1,2,0,3,4,6,7**; ε_4]};
$\boldsymbol{P'} \cap^* \boldsymbol{P''}$ = {[**3,2,4,5,5,4,3,2,3,1**; ε_2],
[**5,3,4,6,5,0,3,4,5,5;** ε_3]};
$\neg^*\boldsymbol{P'} = (C' \setminus \boldsymbol{F'}) \times \{\varepsilon_4\}$;
$\neg^*\boldsymbol{P''} = (C'' \setminus \boldsymbol{F''}) \times \{\varepsilon_5\}$ •

Definition. Fuzzy relations whose credibility levels are given in symbolic form with assumed inequalities instead of their exact numerical values are below called non-deterministic relations.

It is important to remark that in the non-deterministic relation case for finding the most credible algebraic combination of relation instances (e.g. their intersection) no exact numerical credibility values excepting assumed inequalities between them are needed.

III. INFERENCE RULES BASED ON NON-DETERMINED RELATIONS

According to the definition of relative relations the following identities hold:

$$(\boldsymbol{F'}|\,\boldsymbol{F''})\ \bar{\cap}\ \boldsymbol{F''} \equiv \boldsymbol{F'}\ \bar{\cap}\ \boldsymbol{F''} \equiv (\boldsymbol{F''}|\boldsymbol{F'})\ \bar{\cap}\ \boldsymbol{F'}. \quad (11)$$

Assuming that $\boldsymbol{F'}$ and $\boldsymbol{F''}$ are two processes described, respectively, on discrete time-intervals $I' = [i_1,...,i_2]$, $I'' = [i_2,...,i_3]$ where the time-instants $i_1,\ldots,i_2-1$ belong to the "past", i_2 belongs to the (common for I' and I'') "present time" and $i_2+1,\ldots,i_3$ belong to the "future" it will be considered a problem of extrapolation of the process on I'' under the assumption that its behavior in the time-interval I'' has been observed. No special assumptions about the nature of the instant states of the process are made (the processes are thus not obviously numerical) excepting that the $U^{(i)}$ for all i are some isomorphic finite sets, $|U^{(i)}| = K$, K being a natural number. For credibility levels of the trajectories of $\boldsymbol{F'}$ and $\boldsymbol{F''}$ assessment it will be introduced a set of symbolic parameters $\mathbf{e} = [\varepsilon_0, \varepsilon_1,\ldots, \varepsilon_K]$. The components of $\mathbf{e}$ in K! ways can be linearly semi-ordered so that the component ε_0 remains at the first position. Moreover, for any fixed ordering each symbol (excepting the ε_0 one) can be preceded by = or < which gives 2^K different possibilities. Totally, it follows that there are $2^K \cdot$ K! different ways of possible relative credibility levels establishing in this given case.

Example 4

An example of semi-ordering of credibility levels can be presented in the form of a linear graph as shown in Fig. 2.

a/ $\varepsilon_0 = \varepsilon_4 < \varepsilon_5 < \varepsilon_7 < \varepsilon_6 < \varepsilon_{12} < \varepsilon_1 = \varepsilon_3 < \varepsilon_{10} < \varepsilon_2 = \varepsilon_9 = \varepsilon_{11} < \varepsilon_8$

b/

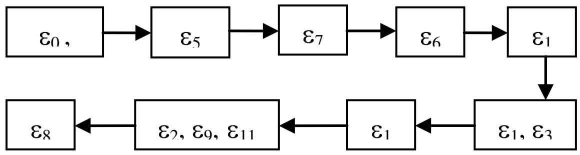

Fig. 2. Example of semi-ordering of credibility levels: a/ algebraic form, b/ graphical form.

For process extrapolation relative credibility levels of longer trajectories should be established [8]. Let us assume that it is given a non-deterministic relation **Y** describing a discrete process in the time-interval $I' \cup I''$. The credibility levels of its instances have been evaluated on the basis of past observation of the process. Therefore, **Y** can play the role of an experimental model of the process, suitable for process prediction decisions making (like training sets used in pattern recognition). For process extrapolation purposes **Y** will be presented in the form of an extended intersection of sub-relations:

$$\mathbf{Y} = \mathbf{Y}_{I'} \cap^* \mathbf{Y}_{I''} \equiv \boldsymbol{Y'} \cup^* \boldsymbol{Y''} \quad (12)$$

where $\boldsymbol{Y'}$ (a sub-process obtained by projection of **Y** on I') is considered as a model of the "past" and, similarly, $\boldsymbol{Y''}$ is used as a model of the "future". The credibility levels assigned to the instances of **Y** are without changing assigned to the corresponding instances of $\boldsymbol{Y'}$ and $\boldsymbol{Y''}$.

Similarly, conditional relations $\boldsymbol{Y'}\,|\boldsymbol{Y''}$ and $\boldsymbol{Y''}|\boldsymbol{Y'}$ can be calculated. The relation $\boldsymbol{Y''}|\boldsymbol{Y'}$ describing future trajectories of the process under the assumption that the past trajectories of the process are known is for process extrapolation of particular interest. Let $\mathbf{x} \in C'$ be a new past trajectory. Then, two cases should be considered:

1st it exists in $\boldsymbol{Y'}$ a trajectory $\mathbf{u}^{(\mathbf{x})}$ identical to **x**:

$$\mathbf{u}^{(\mathbf{x})} \equiv \mathbf{x}; \quad (13)$$

2nd no such trajectory in $\boldsymbol{Y'}$ exists.

Direct one-step extrapolation. In the first case, an intersection:

$$\boldsymbol{Y''}(\mathbf{x}) = \mathbf{u}^{(\mathbf{x})} \cap^* (\boldsymbol{Y''}|\boldsymbol{Y'}) \quad (14)$$

consisting of all sub-trajectories in I'' being prolongations of **x** can be considered. This is illustrated by a graph shown in Fig. 4. The top-row of nodes represent the sub-trajectories belonging to $\boldsymbol{Y'}$ while the bottom-row corresponds to those belonging to their prolongations in the conditional relation $\boldsymbol{Y''}|\boldsymbol{Y'}$. A black node represents the condition $\mathbf{u}^{(\mathbf{x})}$ while grey nodes correspond to its possible prolongations. Moreover, to each sub-trajectory a credibility level is assigned. Therefore, a partial process consisting of the most credible sub-trajectories can be selected according to the rule:

$$z^* = [\boldsymbol{u}^{(x)}, \boldsymbol{v}^{(p)}, u^*] \text{ where } (\boldsymbol{v}^{(p)}: e^{(p)} = max_{(q)}\{e^{(q)}\}, \boldsymbol{v}^{(q)} \in \boldsymbol{Y}''|\boldsymbol{u}^{(x)}, u^* \in \boldsymbol{v}^{(q)}) \quad (15)$$

$e^{(q)}$ being relative credibility levels assigned to the sub-trajectories $\boldsymbol{v}^{(q)}$; u^* denotes a final state of the most credible future sub-trajectory. Bold arrow in Fig. 3 represents the extension of the highest credibility level.

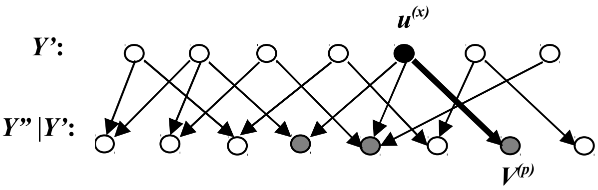

Fig. 3. Direct one-step extrapolation: the sub-trajectories are represented by the nodes.

Indirect one-step extrapolation. The 2nd case of a lack in $\boldsymbol{Y}'$ of an adequate past trajectory needs a more sophisticated approach. Solution of the problem on a general *case-based* approach consisting in finding a closely similar situation is possible. For this purpose a *relative similarity* $\rho(\boldsymbol{u}^{(i)}, \boldsymbol{u}^{(j)})$ of the pairs of elements of the sub-universe C' should be defined. Formally, it can be presented in equivalent form of a parametric relation:

$$\boldsymbol{\rho} \subseteq C' \times C' \times \boldsymbol{e} \quad (16)$$

where $\boldsymbol{e}$ is a finite set of symbols of relative credibility levels on which a semi-ordering relation σ has been imposed. In practice, the semi-ordering σ may be connected with a *similarity measure* of the pairs of sub-trajectories $(\boldsymbol{u}^{(i)}, \boldsymbol{u}^{(j)}) \in C' \times C'$.

Let $\boldsymbol{x} \in C'$ be a given past sub-trajectory. Then it can be defined a partial relation $\boldsymbol{S}_{x^*} \subset \boldsymbol{\rho}$ consisting of all instances containing the pairs $[\boldsymbol{x}, \boldsymbol{u}^{(j)}]$ and their corresponding credibility levels $e_{x,j}$. It will be taken into consideration an intersection:

$$\boldsymbol{Y}'_x = \boldsymbol{S}_{x^*} \cap^* \boldsymbol{Y}' \quad (17)$$

It evaluates the (on similarity measure based) credibility levels of the trajectories $\boldsymbol{u}^{(j)} \in \boldsymbol{Y}'$ used as models approximating the given past sub-trajectory $\boldsymbol{x}$. Then, an extrapolation of $\boldsymbol{x}$ can be reached like in the above described direct one-step extrapolation case when $\boldsymbol{Y}'$ is replaced by $\boldsymbol{Y}'_x$. Analytically, the solution is given by the decision rule:

$$z^* = [\boldsymbol{x}, \boldsymbol{u}^{(j)}, \boldsymbol{v}^{(p)}, u^*] \text{ where } (\boldsymbol{u}^{(j)}, \boldsymbol{v}^{(p)}): e^{(j,p)} = max_{\{j,q\}} min(e_{x,j}, e_{j,q}), (\boldsymbol{x}, \boldsymbol{u}^{(j)}) \in \boldsymbol{Y}'_x, \boldsymbol{v}^{(q)} \in \boldsymbol{Y}''|\boldsymbol{Y}'_x; u^* \in \boldsymbol{v}^{(p)}; e_{x,j}, e_{j,q} \in \boldsymbol{e}. \quad (18)$$

A scheme of this problem solution is shown in Fig. 4. It is important to remark that the most credible solution of the in-direct extrapolation problem due to the relative, non-numerical credibility levels has been found.

Direct multi-step extrapolation. In the one-step extrapolation an extrapolation range is by the length of sub-trajectories of $\boldsymbol{Y}''$ limited. However, it by a multi-step extrapola-

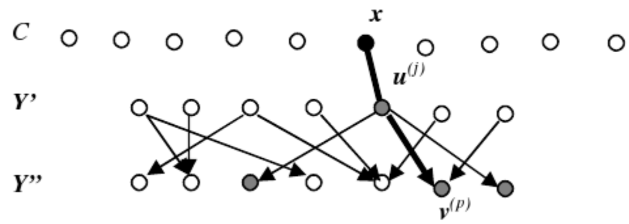

Fig.4. A scheme of in-direct one-step extrapolation problem solution.

tion procedure can be extended. For this purpose a sequence of equal-length overlapping time-intervals $I^{(m)}, \ldots, I^{(2)}, I^{(1)}$ will be considered, each time-interval being divided by a "present" time-instant into its "preceding" and "following" sub-interval, as shown in Fig. 5.

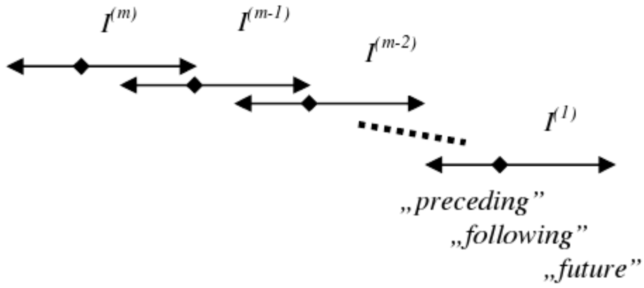

Fig. 5 Extension of process extrapolation intervals.

Let us assume that on the time-intervals the corresponding sub-processes $\boldsymbol{Y}^{(m)}, \boldsymbol{Y}^{(m-1)}, \ldots, \boldsymbol{Y}^{(1)}$ described by isomorphic relations have been defined. In each sub-process $\boldsymbol{Y}^{(\mu)}$ a "preceding" $\boldsymbol{Y}^{(\mu)}{}'$ and "following" $\boldsymbol{Y}^{(\mu)}{}''$ part so that

$$\boldsymbol{Y}^{(\mu)} = \boldsymbol{Y}^{(\mu)}{}' \cap^* \boldsymbol{Y}^{(\mu)}{}'' \quad (19)$$

will be distinguished. Then, an intersection:

$$\boldsymbol{Y} = \boldsymbol{Y}^{(m)} \cap^* \boldsymbol{Y}^{(m-1)} \cap^* \ldots \cap^* \boldsymbol{Y}^{(1)} \quad (20)$$

will be taken into consideration. In the process $\boldsymbol{Y}$ a sub-process $\boldsymbol{Y}^{(m)}{}'$ as the "past" and $\boldsymbol{Y}|\boldsymbol{Y}^{(m)}{}'$ as the "future" can be considered. Formally, solution of the multi-step extrapolation problem is similar to this of the one-step extrapolation one excepting that the "future" sub-universe becomes larger as containing more alternating sub-trajectories leading from a "present" to a "final" state. A solution of the direct multi--step extrapolation problem is given by the decision rule:

$$z^* = [\boldsymbol{u}^{(x)}, \boldsymbol{v}^{*(p)}, u^*] \text{ where } (\boldsymbol{v}^{*(p)}: e^{(p)} = max_{(q)} min\{e^{(q1)}, e^{(q2)}, \ldots, e^{(qm)}\}, \boldsymbol{v}^{*(q)} \in \boldsymbol{Y}''|\boldsymbol{u}^{(x)}) \quad (21)$$

Above, $\boldsymbol{v}^{*(q)}$ denotes the q-th path leading from $\boldsymbol{u}^{(x)}$, $\boldsymbol{u}^{(x)} \in \boldsymbol{Y}^{(m)}{}'$, through a chain of connected sub-trajectories $\boldsymbol{v}^{(q1)}, \boldsymbol{v}^{(q2)}, \ldots, \boldsymbol{v}^{(qm)}$ to a final future state $u^* \in \boldsymbol{v}^{(qm)}$.

Example 5

Systolic pressures of a patient have been daily recorded for several weeks (a real medical record has been used as a basis of this example). The scale of pressure have been divided into intervals as follows:

D_1: 101 – 105 mmHg
D_2: 106 – 110 mmHg,
D_3: 111 – 115 mmHg,
…
D_{24}: 215 – 220 mmHg.

An observed discrete process trajectory after quantization took the form:

$D_5\ D_4\ D_5\ D_4\ D_4\ D_1\ D_6\ D_2\ D_6\ D_6\ D_5\ D_7\ D_4\ D_4\ D_5\ D_6\ D_6\ D_5\ D_3\ D_6 \ldots$

The trajectory has been used as an experimental model of the process. From the above-given sequence of data 16 sub-sequences have been selected as follows:

P_1: $D_5\ D_4\ D_5 \mid D_4\ D_4$,
P_2: $D_4\ D_5\ D_4 \mid D_4\ D_1$,
P_3: $D_5\ D_4\ D_4 \mid D_1\ D_6$,
…
P_{16}: $D_6\ D_6\ D_5 \mid D_3 D_6$

where the marks | separate the (arbitrarily chosen) "past" and "future" parts of the sub-sequences. Then, there have been established formally admissible connections between the sub-sequences, like:

P_1: $D_5\ D_4\ D_5\ \mathbf{D_4\ D_4}$,
P_4: $\mathbf{D_4\ D_4}\ D_1\ D_6\ D_2$
$P_1 – P_4$: $D_5\ D_4\ D_5\ \mathbf{D_4\ D_4}\ D_1\ D_6\ D_2$.

This led to a list of admissible connections shown below:

$\underline{P_1 – P_4}$, $P_1 – P_{13}$, $\underline{P_2 – P_5}$, $\underline{P_3 – P_6}$, $\underline{P_4 – P_7}$, $\underline{P_5 – P_8}$, $\underline{P_6 – P_9}$, $P_6 – P_{16}$, $\underline{P_7 – P_{10}}$,
$P_7 – P_{17}$, $\underline{P_8 – P_{11}}$, $\underline{P_9 – P_{12}}$, $P_{10} – P_4$, $\underline{P_{10} – P_{13}}$, $P_{11} – P_2$, $\underline{P_{11} – P_{14}}$, $\underline{P_{12} – P_{15}}$,
$P_{13} – P_9$, $\underline{P_{13} – P_{16}}$, $P_{14} – P_{10}$, $\underline{P_{14} – P_{17}}$, $\underline{P_{15} – P_{18}}$, $\underline{P_{16} – P_{19}}$, $\underline{P_{17} – P_{20}}$,
$P_{18} – P_4$, $P_{18} – P_{13}$, $P_{20} – P_{17}$…

To the connection their relative credibility levels e can be assigned according to the number of cases they have been really noticed. A symbolic credibility level e_1 is assigned to the connections that have been noticed and e_0 to those which only formally are possible. It is assumed that $e_0 < e_1$; the connections to which e_1 has been assigned are in the list underlined.

Let us assume that an initial sub-process $\boldsymbol{Y'} = [D_5\ D_4\ D_5\}$ has been observed. The question is: what are the most credible extrapolations of the process? Its extrapolations are:

1st step: $\underline{\boldsymbol{Y'} – P_1} = [D_5\ D_4\ D_5 \mid D_4\ D_4\}$ with credibility level e_1;

2nd step: $\underline{\boldsymbol{Y'} – P_1 – P_4} = [D_5\ D_4\ D_5 \mid D_4\ D_4\ D_1\ D_6\ D_2]$ with credibility level $e = \min(e_1,e_1) = e_1$;

$$\underline{\boldsymbol{Y'} – P_1} – P_{13} = [D_5\ D_4\ D_5 \mid D_4\ D_4,\ D_5\ D_6\ D_6]$$

with credibility level $e = \min(e_1,e_0) = e_0$.

In similar way, further process extrapolation can be reached •

IV. Conclusions

Human natural ability to forecast possible future events is based on experience and intuition rather than on formal logical inference. Early computer-aided forecasting systems tried to reach similar goals using formal tools like stochastic processes theory, logical inference methods, etc. However, it occurred that including less rigid, heuristic methods into forecasting procedures may also lead to their improvement. The case-based methods of processes extrapolation as well as relative credibility levels of their future states forecasting presented in this paper belong to this new approach to human thinking imitation. Presentation of a process as a relation makes consideration of not only numerical but also qualitative and multi-aspect processes possible. Credibility levels are considered as comparative, not obviously absolute numerical values. As a consequence, indication of the most credible future states of an extrapolated process is based on a simple comparative analysis instead of rigid numerical calculi. Evidently, some shortcomings with the above-presented approach, like: dependence of the extrapolation effectiveness on the representativeness of the past process observations as models of the future process behavior also exist. The methods of relative credibility levels of process instances assessment need thus deeper investigation which should be a task for future works to be undertaken..

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