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Variable Precision Fuzzy Rough Set Based on Relative Cardinality

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***Abstract*—The fuzzy rough set approach (FRSA) is a theoretical framework that can deal with data analysis of possibilistic information systems. While a set of comprehensive rules can be induced from a possibilistic information system by using FRSA, generation of several intuitively justified rules is sometimes blocked by objects that only partially satisfy the antecedents of the rules. In this paper, we use the variable precision models of FRSA to cope with the problem. The models admit rules that are not satisfied by all objects. It is only required that the proportion of objects satisfying the rules must be above a threshold called a a precision level. In the presented models, the proportion of objects is represented as a relative cardinality of a fuzzy set with respect to another fuzzy set. We investigate three types of models based on different definitions of fuzzy cardinalities including Σ-counts, possibilistic cardinalities, and probabilistic cardinalities; and the precision levels corresponding to the three types of models are respectively scalars, fuzzy numbers, and random variables.**

***Index Terms*—fuzzy set; rough set; variable precision rough set; fuzzy cardinality**

I. Introduction

THE ROUGH set theory proposed by [14] provides an effective tool for extracting knowledge from information systems. A strong assumption about information systems is that each object takes exactly one value with respect to an attribute. However, in practice, we may only have incomplete information about the values of an object's attributes. Thus, more general information systems have been introduced to represent incomplete information ([7], [8], [13], [17], [19]), whereas fuzzy rough set theory proposed in [5] has been considered as an important mathematical tool to deal with rough set-based analysis of possibilistic information systems. However, rough set analysis is notoriously sensitive to noisy information, because a mistakenly labeled sample may deteriorate the quality of approximations significantly.

The variable precision rough set (VPRS) theory introduced in [6], [23] is a main approach to improve the robustness of rough set analysis. In VPRS, classification rules can be induced even though not satisfied by all objects. It is only required that the proportion of objects satisfying the rules must be above a threshold called a precision level. The idea of error-tolerance by variable precision has been applied to fuzzy rough set theory [1], [2], [9]–[12], [18]. In this paper, we extend the previous work on variable precision fuzzy rough set (VPFRS) theory by using the notion of fuzzy cardinality. The proportion of objects satisfying a rule is modeled as a relative fuzzy cardinality in our approach. Because a fuzzy cardinality may be a scalar, a fuzzy number, or a random variable, we can induce three types of models depending on what kinds of fuzzy cardinalities are taken as the precision levels.

The remainder of the paper is organized as follows. In Section II and III, we review rough set theory and the notion of fuzzy cardinality respectively. In Section IV, we introduce three types of VPFRS models based on the relative cardinalities of fuzzy sets. Section V contains some concluding remarks.

II. Rough Set Theory

A. Classical rough set

The basic construct of rough set theory is an *approximation space*, which is defined as a pair (U, R), where U is a finite universe and $R \subseteq U \times U$ is an equivalence relation on U. We write an equivalence class of R as $[x]_R$ if it contains the element x. For any subset X of the universe, the lower approximation and upper approximation of X are defined as follows:

$$\underline{R}X = \{x \in U \mid \forall y((x, y) \in R \rightarrow y \in X)\}, \quad (1)$$

$$\overline{R}X = \{x \in U \mid \exists y((x, y) \in R \wedge y \in X)\}. \quad (2)$$

This definition of rough set is called the *logic-based* definition [1].

An alternative way to define rough sets is to use the *rough membership function* [16]. Given an approximation space (U, R) and a subset $X \subseteq U$, the rough membership function $\nu_X^R : U \rightarrow [0, 1]$ is defined as

$$\nu_X^R(x) = \frac{|[x]_R \cap X|}{|[x]_R|}. \quad (3)$$

The value $\nu_X^R(x)$ is interpreted as the degree that x belongs to X in view of knowledge about x expressed by the indiscernibility relation R or the degree to which the R-equivalence class $[x]_R$ is included in the set X. Then, the lower approximation and upper approximation of X are defined as follows:

$$\underline{R}X = \{x \in U \mid \nu_X^R(x) = 1\}, \tag{4}$$

$$\overline{R}X = \{x \in U \mid \nu_X^R(x) > 0\}. \tag{5}$$

This definition of rough set is called the *frequency-based* definition [1].

Although an approximation space is an abstract framework used to represent classification knowledge, it can easily be derived from a concrete information system. Pawlak ([15]) defined an information system[1] as a tuple $T = (U, A, \{V_i \mid i \in A\}, \{f_i \mid i \in A\})$, where U is a nonempty finite set, called the universe; A is a nonempty finite set of primitive attributes; for each $i \in A$, V_i is the domain of values of i; and for each $i \in A$, $f_i : U \to V_i$ is a total function. In decision analysis, we assume the set of attributes is partitioned into $\{d\} \cup (A - \{d\})$, where d is called the *decision attribute*, and the remaining attributes in $C = A - \{d\}$ are called *condition attributes*. Given a subset of attributes B, the *indiscernibility relation* with respect to B is defined as $ind(B) = \{(x, y) \mid x, y \in U, f_i(x) = f_i(y) \forall i \in B\}$. Obviously, for each $B \subseteq A$, $(U, ind(B))$ is an approximation space.

B. Fuzzy rough set

In [5], it is shown that fuzzy sets and rough sets are essentially different but complementary for the modeling of uncertainty. Despite the essential difference between fuzzy sets and rough sets, there are approaches to incorporate the notion of fuzzy sets into rough set models [5]. One approach is to consider the lower and upper approximations of a fuzzy concept in an approximation space, which results in the *rough fuzzy set*. The other approach is to consider the approximations of a crisp or fuzzy concept in a *fuzzy approximation space*, which is defined as a pair (U, R), where R is a fuzzy binary relation on U, i.e., $R : U \times U \to [0, 1]$. This leads to the *fuzzy rough set*. Let (U, R) be a fuzzy approximation space and let X be a fuzzy subset of U with membership function $\mu_X : U \to [0, 1]$. Then, $\underline{R}X, \overline{R}X : U \to [0, 1]$ are defined by

$$\underline{R}X(x) = \inf_{y \in U} R(x, y) \to_\otimes \mu_X(y), \tag{6}$$

$$\overline{R}F(x) = \sup_{y \in U} R(x, y) \otimes \mu_X(y), \tag{7}$$

where $\otimes : [0, 1] \times [0, 1] \to [0, 1]$ is a t-norm and $\to_\otimes : [0, 1] \times [0, 1] \to [0, 1]$ is an implication with respect to $\otimes$. There are several different definitions of implication functions $\to_\otimes$ for a given t-norm, including the S-implication defined by $a \to_\otimes b = 1 - (a \otimes (1 - b))$ and the R-implication defined by $a \to_\otimes b = \sup\{c \mid a \otimes c \leq b\}$.

[1]Also called knowledge representation systems, data tables, or attribute-value systems

C. Variable precision rough set

By using the frequency-based definition of the rough set, an object x belongs to the lower approximation of a set X if its rough membership value is 1, i.e., $x \in \underline{R}X$ iff $\nu_X^R(x) = 1$. However, the requirement seems overly strict since in the noisy environment, it may be difficult to require an R-equivalence class $[x]_R$ is totally included in a set X. The purpose of variable precision rough set (VPRS) theory is to address the issue by relaxing the strict requirement of total inclusion to partial inclusion. Technically, this is achieved by two parameters called precision levels. Let l and u be real numbers such that $0 \leq l < u \leq 1$. Then, the u-lower approximation and the l-upper approximation of X are defined as follows:

$$\underline{R}_u X = \{x \in U \mid \nu_X^R(x) \geq u\}, \tag{8}$$

$$\overline{R}_l X = \{x \in U \mid \nu_X^R(x) > l\}. \tag{9}$$

III. Fuzzy Cardinality

The cardinality of a fuzzy set is normally used to evaluate fuzzy quantified sentences. For example, to evaluate the truth degree of the sentence "Most students are young," we have to determine if the cardinality of the set of young students satisfies the interpretation of the fuzzy quantifier "most." In applications, two kinds of cardinality are considered: *absolute cardinality*, which measures the number of elements in a set; and *relative cardinality*, which measures the percentage of elements of one set (called the referential set) that are also present in another set ([4]). Since VPRS is concerned with the percentage of elements in an object's indiscernibility class that are also in the approximated concept, relative cardinality plays an important role in our analysis.

Several approaches for measuring the cardinality of a fuzzy set have been proposed in the literature. The approaches, which extend the classic approach in different ways, can be classified into two categories: *scalar cardinality* approaches and *fuzzy cardinality* approaches. The former measure the cardinality of a fuzzy set by means of a scalar value, either an integer or a real value; whereas the latter assume that the cardinality of a fuzzy set is just another fuzzy set over the non-negative numbers ([4]). The most simple scalar cardinality of a fuzzy set is its *power* (also called the Σ-count), which is defined as the summation of the membership degrees of all elements ([3]). Formally, for a given fuzzy subset F on the universe U, the Σ-count of F is defined as

$$\Sigma\sharp(F) = \sum_{x \in U} \mu_F(x). \tag{10}$$

The relative cardinality of a fuzzy set G with respect to another fuzzy set F is then defined as ([20]):

$$\Sigma\sharp(G/F) = \frac{\Sigma\sharp(F \cap G)}{\Sigma\sharp(F)}. \tag{11}$$

Subsequently, [21] proposed a fuzzy subset $Z(F)$ of $\mathbb{N}$ as the measure of the absolute cardinality of a fuzzy set F such that the membership degree of a natural number $k \in \mathbb{N}$ in $Z(F)$ is defined as

$$Z(F,k) = \sup\{\alpha \mid |F_\alpha| = k\}, \tag{12}$$

where F_α is the α-cut of F. In addition, a fuzzy multiset $Z(G/F)$ over $[0,1]$ is introduced in ([22]) to measure the fuzzy relative cardinality of G with respect to F. The membership function of $Z(G/F)$, written in the standard integral notation, is defined as

$$Z(G/F) = \sum_{\alpha\in\Lambda(F)\cup\Lambda(G)} \alpha / \frac{|F_\alpha \cap G_\alpha|}{|F_\alpha|}, \tag{13}$$

where $\Lambda(F)$ and $\Lambda(G)$ are the level sets of F and G respectively, i.e., $\Lambda(F) = \{\mu_F(x) \mid x \in U\}$. Delgado et al. [4] proposed a more compact representation of $Z(G/F)$ by transforming the fuzzy multiset into a fuzzy subset of rational numbers in $[0,1]$. The representation is formulated as follows:

$$ES(G/F,q) = \sup\{\alpha \in \Lambda(G/F) \mid \frac{|(F\cap G)_\alpha|}{|F_\alpha|} = q\} \tag{14}$$

for any $q \in \mathbb{Q} \cap [0,1]$, where $\Lambda(G/F) = \Lambda(F\cap G) \cup \Lambda(G)$.

In the context of a finite universe U, Delgado et al. [4] proposed a family of fuzzy measures $\mathcal{E}$ for absolute cardinalities based on the evaluation of fuzzy logic sentences. To define the measures, the possibility of a fuzzy set F containing at least k elements is identified with the truth degree of the fuzzy sentence $\exists X \subseteq U(|X| = k \wedge X \subseteq F)$, which can be formally defined as

$$L(F,k) = \begin{cases} 1, & \text{if } k = 0, \\ 0, & \text{if } k > |U|, \\ \bigoplus_{X\subseteq_k U} \bigotimes_{x\in X} \mu_F(x), & \text{if } 1 \le k \le |U|, \end{cases} \tag{15}$$

where $X \subseteq_k U$ denotes that X is any k-element subset of U. Then, the possibility that F contains exactly k elements is formulated as follows:

$$E(F,k) = L(F,k) \otimes \neg L(F,k+1), \tag{16}$$

where $\otimes$ is any t-norm (not necessarily the same as that used in the definition of $L(F,k)$), and $\neg$ stands for a fuzzy negation. Each member of the family $\mathcal{E}$ is determined by the choice of s-norm, t-norms and negation in (15) and (16). Using $\max$ and $\min$ in (15) and standard negation as well as Lukasiewicz's t-norm $\max(0, a+b-1)$ in (16), a probabilistic measure of absolute cardinality ED defined as

$$ED(F,k) = \alpha_k - \alpha_{k+1} \tag{17}$$

is shown to be a member of the family $\mathcal{E}$, where α_k is the kth largest value of the multiset $\{\mu_F(x) \mid x \in U\}$ for $1 \le k \le |U|$, $\alpha_0 = 1$, and $\alpha_k = 0$ when $k > |U|$. The relative version of ED is also defined as

$$ER(G/F,q) = \sum_{i:\frac{|(F\cap G)_{\alpha_i}|}{|F_{\alpha_i}|}=q} (\alpha_i - \alpha_{i+1}) \tag{18}$$

for any $q \in \mathbb{Q} \cap [0,1]$, where α_i is the ith largest value of $\Lambda(G/F)$.

IV. Variable Precision Fuzzy Rough Set

The main feature of VPRS is to allow objects that partially violate the indiscernibility principle. For example, if an object belongs to the lower approximation of a set, it is not necessary that all objects that are indiscernible with the object belong to the target set. Instead, the only requirement for an object to be included in the lower approximation of a target set is that a sufficiently large portion of the object's indiscernibility class belongs to the target set. From the viewpoint of the logic-based definition, this amounts to relax the universal quantifier in (1) and strengthen the existential quantifier in (2) to a proportional quantifier determined by u and l respectively. On the other hand, in the frequency-based definition, the lower approximation and the upper approximation of X correspond to the 1-cut and strict 0-cut of the rough membership function respectively[2]. Therefore, from the viewpoint of the frequency-based definition, VPRS simply decreases the cutting point from 1 to u for the lower approximation and increases the cutting point from 0 to l for the upper approximation.

Since the logic-based definition and frequency-based definition are equivalent for classical rough set, the changes of quantifiers or cutting points lead to the same VPRS model. However, for the fuzzy rough set, the situation is quite different. The definition of fuzzy rough approximations in (6) and (7) is essentially a generalization of the logic-based definition of the classical rough set. By using the notion of relative cardinality, we can now present the frequency-based definition of fuzzy rough set.

Let (U,R) be a fuzzy approximation space and let X be a fuzzy subset of U with membership function $\mu_X : U \to [0,1]$. Then, since R is a fuzzy relation, the indiscernibility class of an object x is a fuzzy subset with membership function $R(x) : U \to [0,1]$ such that $R(x)(y) = R(x,y)$ for every $y \in U$. Thus, we can define the *fuzzy-rough membership function* $\nu_X^R : U \to [0,1]$ as follows

$$\nu_X^R(x) = \phi(X/R(x)), \tag{19}$$

where ϕ may be any one of the relative cardinality $\Sigma\sharp$, ES, or ER defined in Section III. Clearly, we can no longer define $\underline{R}(X)$ and $\overline{R}(X)$ as cut sets of the fuzzy-rough membership function as in the case of classical rough set. Hence, the logic-based and frequency-based definitions of the fuzzy rough set are not equivalent any more. Consequently, the VPFRS models for these two kinds of fuzzy rough sets should be also significantly different.

There have been several VPFRS models that are derived from modifying the logic-based definition of the fuzzy rough set [2], [9]–[12], [18]. However, the frequency-based approach to VPFRS remains largely unexplored except the vaguely quantified rough set (VQRS) approach [1]. The main idea of VQRS is that the membership degree of an object in the lower and upper approximations of a target set is determined

[2]Recall that the α-cut and the strict α-cut of a membership function $\nu : U \to [0,1]$ are defined as $\{x \in U \mid \nu(x) \ge \alpha\}$ and $\{x \in U \mid \nu(x) > \alpha\}$ respectively.

by applying fuzzy quantifiers, such as "most" and "some", to the object's fuzzy-rough membership degree in the target set. Nevertheless, the fuzzy-rough membership function used in VQRS is simply the $\Sigma\sharp$ relative cardinality. Therefore, the purpose of this paper is to investigate frequency-based VPFRS models by using different fuzzy-rough membership functions. In addition, we still use precision levels to define VPFRS instead of the fuzzy quantifiers in VQRS. In this regard, we can consider three kinds of generalizations of VPRS to VPFRS.

First, if the scalar precision levels $l \in [0, 0.5]$ and $u \in (0.5, 1]$ are given, then we use the relative Σ-count to measure if an object satisfies the partial precision requirement. Thus, the u-lower approximation and the l-upper approximation of X are defined as subsets of U as follows:

$$\underline{R}_u(X)(x) = \{x \in U \mid \Sigma\sharp(X/R(x)) \geq u\}, \quad (20)$$

$$\overline{R}_l(X)(x) = \{x \in U \mid \Sigma\sharp(X/R(x)) > l\}. \quad (21)$$

Second, if the precision levels are fuzzy numbers, then we use the relative cardinality ES to measure an object's fuzzy-rough membership degree. Let $\tilde{l} : [0, 0.5] \cap \mathbb{Q} \to [0, 1]$ and $\tilde{u} : (0.5, 1] \cap \mathbb{Q} \to [0, 1]$ be two fuzzy numbers. Then, $\tilde{u}$-lower approximation and the $\tilde{l}$-upper approximation of X are defined as follows:

$$\underline{R}_{\tilde{u}}(X)(x) = \pi(ES(X/R(x)) \geq \tilde{u}), \quad (22)$$

$$\overline{R}_{\tilde{l}}(X)(x) = \pi(ES(X/R(x)) > \tilde{l}), \quad (23)$$

In the above definition, the relative cardinality ES is regarded as a fuzzy number, and $\pi(\cdot)$ returns the possibility of the comparison statement between two fuzzy numbers based on the extension principle. For example, the possibility of a fuzzy number $\tilde{l_1}$ being greater than another fuzzy number $\tilde{l_2}$ is defined as $\pi(\tilde{l_1} > \tilde{l_2}) = \sup_{x_1 > x_2} \min(\mu_{\tilde{l_1}}(x_1), \mu_{\tilde{l_2}}(x_2))$.

Third, if the precision levels are random variables, we can use the relative cardinality ER to represent an object's fuzzy-rough membership degree. Let us overload the notation $ER(G/F)$ to denote a $[0, 1] \cap \mathbb{Q}$-valued random variable whose probability mass function is defined as $Pr(ER(G/F) = q) = ER(G/F, q)$ for any fuzzy sets F and G, and let $\hat{l}$ and $\hat{u}$ be $[0, 0.5] \cap \mathbb{Q}$-valued and $(0.5, 1] \cap \mathbb{Q}$-valued random variables respectively. Then, the $\hat{u}$-lower approximation and the $\hat{l}$-upper approximation of X are defined as follows:

$$\underline{R}_{\hat{u}}(X)(x) = Pr(ER(X/R(x)) \geq \hat{u}), \quad (24)$$

$$\overline{R}_{\hat{l}}(X)(x) = Pr(ER(X/R(x)) > \hat{l}). \quad (25)$$

The three types of VPFRS models are called VPFRS1, VPFRS2, and VPFRS3 respectively. As a scalar can be regarded as a single-point (possibility or probability) distribution, all three types of models are applicable when the precision level is a scalar. A typical application of VPFRS2 is when the precision levels are given by linguistic terms. For example, it may be required that the precision level is moderately high. On the other hand, VPFRS3 may be applied when the precision level is set as a sub-interval of $(0.5, 1] \cap \mathbb{Q}$. In this case, the precision level is regarded as a uniform distribution on the sub-interval, so it is actually a random variable.

V. Concluding Remarks

In this paper, we propose three types of frequency-based VPFRS models to improve the robustness of the fuzzy rough set. As rough set analysis is sensitive to noisy samples, VPRS can avoid the problem by tolerating partially inconsistency data. The transition from classical rough set theory to VPRS can be achieved by generalizing the quantifiers in the logic-based definition or by changing the cutting points of the rough membership function. Although these two approaches result in the same VPRS model due to the equivalence between logic-based and frequency-based definitions of classical rough set, the situation become radically different in the case of fuzzy rough set. Since the logic-based definition of fuzzy rough set can not be derived from the fuzzy-rough membership function, the VPFRS models obtained from modifying the former should be significantly different from those based on the latter. While most existing VPFRS models adopt the modification of the logic-based definition, the VQRS based on the fuzzy-rough membership function is also shown to outperform the original fuzzy rough set in a benchmark data set [1]. However, since VQRS only consider the fuzzy-rough membership function based on the relative Σ count, it does not utilize the full potential of the notions of fuzzy cardinalities. It has been also suggested that fuzzy numbers are more appropriate than scalars as cardinalities of fuzzy sets [4]. Thus, our work extends the VQRS model by using different types of relative cardinalities to define the fuzzy-rough membership function. This approach provides a greater flexibility to specify the precision levels when the parameters of the VPFRS models can not be determined precisely, since we allow the precision levels to be scalars, linguistic terms, or random variables.

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References

[1] C. Cornelis, M. De Cock, and A.M. Radzikowska. Vaguely quantified rough sets. In *Proceedings of the 11th International Conference Rough Sets, Fuzzy Sets, Data Mining and Granular Computing (RSFDGrC)*, volume 4482 of *LNCS*, pages 87–94. Springer, 2007.

[2] C. Cornelis, N. Verbiest, and R. Jensen. Ordered weighted average based fuzzy rough sets. In *Proceedings of the 5th International Conference on Rough Set and Knowledge Technology (RSKT)*, volume 6401 of *LNCS*, pages 78–85. Springer, 2010.

[3] A. de Luca and S. Termini. A definition of a nonprobabilistic entropy in the setting of fuzzy sets theory. *Information and Control*, 20(4):301–312, 1972.

[4] M. Delgado, D. Sánchez, M.J. Martín-Bautista, and M.A. Vila. A probabilistic definition of a nonconvex fuzzy cardinality. *Fuzzy Sets and Systems*, 126(2):177–190, 2002.

[5] D. Dubois and H. Prade. Rough fuzzy sets and fuzzy rough sets. *International Journal of General Systems*, 17(2-3):191–209, 1990.

[6] J.D. Katzberg and W. Ziarko. Variable precision extension of rough sets. *Fundamenta Informaticae*, 27(2/3):155–168, 1996.

[7] M. Kryszkiewicz. Properties of incomplete information systems in the framework of rough sets. In L. Polkowski and A. Skowron, editors, *Rough Sets in Knowledge Discovery*, pages 422–450. Physica-Verlag, 1998.

[8] W. Lipski. On databases with incomplete information. *Journal of the ACM*, 28(1):41–70, 1981.

[9] A. Mieszkowicz-Rolka and L. Rolka. Fuzzy implication operators in variable precision fuzzy rough sets model. In *Proceedings of the 7th International Conference on Artificial Intelligence and Soft Computing (ICAISC)*, volume 3070 of *LNCS*, pages 498–503. Springer, 2004.

[10] A. Mieszkowicz-Rolka and L. Rolka. Remarks on approximation quality in variable precision fuzzy rough sets model. In *Proceedings of the 4th International Conference on Rough Sets and Current Trends in Computing*, volume 3066 of *LNCS*, pages 402–411. Springer, 2004.

[11] A. Mieszkowicz-Rolka and L. Rolka. Variable precision fuzzy rough sets. I:144–160, 2004.

[12] A. Mieszkowicz-Rolka and L. Rolka. An approach to parameterized approximation of crisp and fuzzy sets. In *Proceedings of the 5th International Conference on Rough Sets and Current Trends in Computing*, volume 4259 of *LNCS*, pages 127–136. Springer, 2006.

[13] K. Myszkorowski. Multiargument relationships in fuzzy databases with attributes represented by interval-valued possibility distributions. In S.O. Kuznetsov, D. Slezak, D.H. Hepting, and B.G. Mirkin, editors, *Proceedings of the 13th International Conference on Rough Sets, Fuzzy Sets, Data Mining and Granular Computing*, volume 6743 of *LNCS*, pages 199–206. Springer, 2011.

[14] Z. Pawlak. Rough sets. *International Journal of Computer and Information Sciences*, 11(15):341–356, 1982.

[15] Z. Pawlak. *Rough Sets–Theoretical Aspects of Reasoning about Data*. Kluwer Academic Publishers, 1991.

[16] Z. Pawlak and A. Skowron. Rough membership functions. In R. Yager, M. Fedrizzi, and J. Kacprzyk, editors, *Advances in the Dempster-Shafer Theory of Evidence*, pages 251–271. John Wiley&Sons, 1994.

[17] H. Sakai, M. Nakata, and D. Slezak. A prototype system for rule generation in lipskis incomplete information databases. In S.O. Kuznetsov, D. Slezak, D.H. Hepting, and B.G. Mirkin, editors, *Proceedings of the 13th International Conference on Rough Sets, Fuzzy Sets, Data Mining and Granular Computing*, volume 6743 of *LNCS*, pages 175–182. Springer, 2011.

[18] J.M. Fernández Salido and S. Murakami. Rough set analysis of a general type of fuzzy data using transitive aggregations of fuzzy similarity relations. *Fuzzy Sets and Systems*, 139(3):635–660, 2003.

[19] Y.Y. Yao and Q. Liu. A generalized decision logic in interval-set-valued information tables. In N. Zhong, A. Skowron, and S. Ohsuga, editors, *New Directions in Rough Sets, Data Mining, and Granular-Soft Computing*, LNAI 1711, pages 285–293. Springer-Verlag, 1999.

[20] L.A. Zadeh. The concept of a linguistic variable and its applications in approximate reasoning. *Information Sciences*, 8:199–251, 1975.

[21] L.A. Zadeh. A theory of approximate reasoning. In J.E. Hayes, D. Mitchie, and L.I. Mikulich, editors, *Machine Intelligence*, volume 9, pages 149–194, 1979.

[22] L.A. Zadeh. A computational approach to fuzzy quantifiers in natural languages. *Computers and Mathematics with Applications*, 9(1):149–184, 1983.

[23] W. Ziarko. Variable precision rough set model. *Journal of Computer and System Sciences*, 46(1):39–59, 1993.