Collective Belief Revision in Linear Algebra

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Abstract—**Although the logic of belief update has mainly concerned a belief state of one agent thus far, the real world settings require us to implement simultaneous belief changes. Here, however, we need to manage so many indices: agent names, time stamps, and the difference of information. In this paper, we introduce the notation of vectors and matrices for the simultaneous informing action. By this, we show that a matrix can represent a public announcement and/or a consecutive message passing, with the time of the change of belief states properly. A collective belief state multiplied by a communication matrix, including matrices of accessibility in Kripke semantics, becomes a hypercuboid.**

I. Who Knows What at Which Time?

THE authors have tackled legal reasoning system thus far [12], [13], in which one of the main issues is to properly represent *who knows what at which time*. In Fig. 1, we show the informing actions of three agents: judge, lawyer, and the suspected. At the initial stage, the judge sentenced the suspected to be guilty, and at the same time the lawyer pleaded innocent to the judge with a new witness. The judge changed his mind and he was going to sentence the defendant to be innocent, but at the same time the maladjusted defendant insulted the judge in the court, and which badly impressed jury members ...
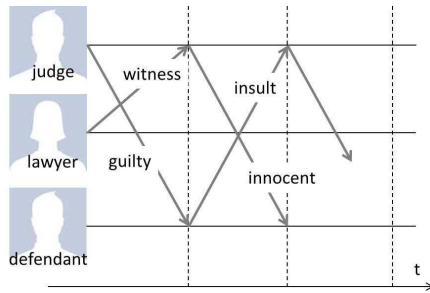

Fig. 1.   Informing Agents on Time-axis

Multi-agents communication includes such many factors as agent ID, many messages, and time. In this paper, we introduce vectors and matrices to represent those agents' informing actions collectively, to clarify the complicated relations of those many indices.

II. Preliminaries

First we show the simplest prescription for an informing action; in a similar way to FIPA/ACL [3], we place the precondition in the upper deck and the result in the lower deck.[1]

$$\frac{B_i\varphi}{B_j\varphi}[c_{ij}^{\varphi}] \tag{1}$$

That is, when agent i believes φ ($B_i\varphi$) and there is a communication from i to j (c_{ij}^{φ}), agent j comes to believe φ ($B_j\varphi$). At this stage, there are several issues we need to consider:

- The problem of belief revision; the recipient of information may not believe what was informed of, or he/she may need to change some of what he/she has believed.
- The resultant state should include nested belief states, i.e., both of the sender and the recipient should recognize that the information is shared between them as $B_iB_j\varphi$ and $B_jB_i\varphi$. In addition, if those agents are quite introspective, each of them also possesses $B_iB_i\varphi$ and $\neg B_i\neg\varphi$.[2]

Incidentally, a Kripke frame is such $\mathfrak{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{V} \rangle$ that $\mathcal{W}$ is a set of possible worlds, $\mathcal{R}$ is the accessibility of belief modal operator B, and $\mathcal{V}$ gives valuation to each φ. Dynamic Epistemic Logic (DEL) [2] presents a change of belief state, restricting accessibility to possible worlds, as:

$$\mathfrak{M}, w \models [\varphi!]\psi \iff \mathfrak{M}^{\varphi!}, w \models \psi. \tag{2}$$

where $R^{\varphi!}$ in $\mathfrak{M}^{\varphi!}$ is:

$$R^{\varphi!}(w) = R(w) \cap \{w' \in W \mid \mathfrak{M}, w' \models \varphi\}.$$

On the contrary, Public Announcement Logic (PAL) [11] masks those contradicting possible worlds, as follows.

$$\mathfrak{M}, w \models [!\varphi]\psi \iff \mathfrak{M}^{[\![\varphi]\!]}, w \models \psi.$$

where in $\mathfrak{M}^{[\![\varphi]\!]}$ let $W_{\uparrow\varphi}$ be the worlds in which φ holds and

$$\mathfrak{M}, w \models \varphi \to w \in V_{W_{\uparrow\varphi}}(\psi).$$

Note that the significance of these methods is to make formula $[!\varphi]B\varphi$ valid since φ holds in all the accessible possible worlds.

Among various trials to represent agent communication formally [1], [9], [10], Yamada [14] showed a command from i to j as $[!_{(i,j)}\chi]$ where χ is the contents of the command. Kobayashi and Tojo [6], [7] generalized this notion to an informing action, representing the dynamic operator as $[inf_{ij}^{\varphi}]$.

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[1] In this paper, we disregard U (uncertain) and U_{if} (uncertain if) operators for simplicity.

[2] In general, belief modality is often implemented with $KD45$, including $B_i\varphi \to B_iB_i\varphi$ (Axiom 4) and $B_i\varphi \to \neg B_i\neg\varphi$ (Axiom D), while knowledge modality requires $KT5$ including $K_i\varphi \to \varphi$ (Axiom T).

As for linear algebraic representation of belief, Fusaoka [4] used a matrix to show probability of knowledge source. Also, as we have mentioned, Liau [8] represented the network of accessibility in matrix. We combined these works, though we avoid probabilistic point of view and restricted the elements to truth values.

III. INFORMING ACTION

The belief modality B_i^t represents the belief state of agent i at time t. As to information φ, the belief states of multiple agents are written collectively in a vector as follows.

$$\begin{pmatrix} B_1^t\varphi \\ B_2^t\varphi \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}$$

where 1/0 are the truth values. Therefore, our specification of belief revision of agent j, with regard to (1), becomes:

$$B_j^{t+1}\varphi = B_j^t\varphi \vee (c_{ij}^\varphi \wedge B_i^t\varphi). \tag{3}$$

Here, c_{ij}^φ represents the informing action from j to i.

We define the addition and the multiplication of linear algebra as the logical 'or' and the logical 'and', respectively, as follows.

$\wedge$	1	0
1	1	0
0	0	0

$\vee$	1	0
1	1	1
0	1	0

Then, we can generalize (3) to be:

$$B_j^{t+1}\varphi = \bigvee_i (c_{ij}^\varphi \wedge B_i^t\varphi), \tag{4}$$

and the dynamic operator becomes such an $n \times n$ matrix that

$$(c_{ij}^\varphi) = \begin{pmatrix} 1 & 0 & \cdots \\ 1 & 1 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

where n is the number of agents. Its (i, j)-element represents the truth value of c_{ij}^φ, that is the existence of informing action from j to i. We assume that diagonal elements c_{ii} ($i = 1, \cdots, n$) are always true to maintain his/her original knowledge as (3). Now, the collective belief revision becomes:

$$\begin{pmatrix} B_1^{t+1}\varphi \\ B_2^{t+1}\varphi \\ \vdots \end{pmatrix} = (c_{ij}^\varphi) \begin{pmatrix} B_1^t\varphi \\ B_2^t\varphi \\ \vdots \end{pmatrix}.$$

Example 1: Let

$$(c_{ij}^\varphi) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} B_1^t\varphi \\ B_2^t\varphi \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

Then,

$$\begin{pmatrix} B_1^{t+1}\varphi \\ B_2^{t+1}\varphi \end{pmatrix} = (c_{ij}^\varphi) \begin{pmatrix} B_1^t\varphi \\ B_2^t\varphi \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Namely, by the informing action from agent 2 to 1, as is $(1, 2)$-element of the matrix, agent 1 comes to know φ. ∎

In the following examples, we highlight our attention with the boxed truth values.

Example 2: Suppose the following three kinds of communication matrices:

$$C_1 = \begin{pmatrix} 1 & \boxed{1} & 0 \\ \boxed{1} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ C_2 = \begin{pmatrix} 1 & \boxed{1} & 0 \\ 0 & 1 & 0 \\ 0 & \boxed{1} & 1 \end{pmatrix}, \ C_3 = \begin{pmatrix} 1 & \boxed{1} & \boxed{1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

If there happened a reciprocal and simultaneous communication as to the same information, the matrix becomes symmetric (C_1), that is, agent 2 tells some information to agent 1 (c_{12}) and at the same time so does agent 1 to agent 2 (c_{21}). An agent can announce some information publicly, in which case a certain column is filled with all 1's (the second column of C_2). If multiple different agents tell the same information to a certain agent, then there appear multiple 1's in the same row; if agent 2 and 3 inform the same content to 1, then the situation becomes C_3. ∎

IV. TRANSITIVE COMMUNICATION

Let us consider to connect two communications. Now, we introduce a vector representation for the collective belief state of multiple agents at time t:

$$B^t\varphi = \begin{pmatrix} B_1^t\varphi \\ B_2^t\varphi \\ \vdots \end{pmatrix}.$$

For the time being, we may omit φ unless we need to mention it explicitly. Two consecutive informing actions can be written in the following matrix multiplication.

$$\begin{pmatrix} B_1 \\ B_2 \\ \vdots \end{pmatrix}^{t+2} = \begin{pmatrix} c'_{11} & c'_{12} & \cdots \\ c'_{21} & c'_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} & \cdots \\ c_{21} & c_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ \vdots \end{pmatrix}^t.$$

A. Associativity

First, we need to prove that communication matrices are associative. Let X and Y be communication matrices and B^t be a collective belief state.

$$(XY)B^t = X(YB^t).$$

As $XY = \bigvee_k (x_{ik} \wedge y_{kj})$,

$$(XY)B^t = \bigvee_l (\bigvee_k (x_{lk} \wedge y_{kl}) \wedge B_l^t) = \bigvee_l \bigvee_k (x_{lk} \wedge y_{kl} \wedge B_l^t).$$

On the other hand, since $YB^t = \bigvee_l (y_{kl} \wedge b_l)$,

$$X(YB^t) = \bigvee_k (x_{lk} \wedge \bigvee_l (y_{kl}) \wedge B_l^t) = \bigvee_k \bigvee_l (x_{lk} \wedge y_{kl} \wedge B_l^t).$$

As the both results meet, Q.E.D.

B. Repetitive communication

Let us consider the case that the same communication matrix is employed repeatedly.

$$B^{t+n} = (c_{ij})^n B^t.$$

Example 3: Suppose

$$C = (c_{ij}) = \begin{pmatrix} 1 & \boxed{1} & 0 \\ 0 & 1 & \boxed{1} \\ 0 & 0 & 1 \end{pmatrix},$$

that is, c_{12} and c_{23} are true, besides self-informing. Then,

$$C^2 = \begin{pmatrix} 1 & 1 & \boxed{1} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Namely, agent 3 is reachable from agent 1 in two steps. ∎

Now let B^t be the initial belief state of the community, and let us consider the following sequence.

$$B^{t+1} = CB^t, \ B^{t+2} = CB^{t+1}, \ B^{t+3} = CB^{t+2}, \ \cdots.$$

Note that the number of 1's in the matrix increases monotonously, since $c_{ii} = 1$ and once an agent believes the proposition (s)he keeps it in his/her recognition. Let B^* be the fixed point such that $B^* = CB^*$. If $B^* = B^{t+k}$, C^k is the transitive closure of the communication graph.

C. Anti-commutativity

As is the case in usual matrix multiplication, communication matrices are not commutative.

Example 4:

$$\begin{pmatrix} 1 & 0 & \boxed{1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 \\ \boxed{1} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are communication matrices from agent 3 to 1 (left), and that from 1 to 2 (right), respectively. If agent 3 first believes φ, as

$$\begin{pmatrix} 1 & 0 & 0 \\ \boxed{1} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & \boxed{1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ \boxed{1} \end{pmatrix} = \begin{pmatrix} \boxed{1} \\ \boxed{1} \\ 1 \end{pmatrix},$$

agent 2 comes to believe φ. But, when agent 1 does not believe φ, as

$$\begin{pmatrix} 1 & 0 & \boxed{1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ \boxed{1} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ \boxed{1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

the first informing action, that is from 1 to 2, results in vain, and thus 2 still remains ignorant of φ. ∎
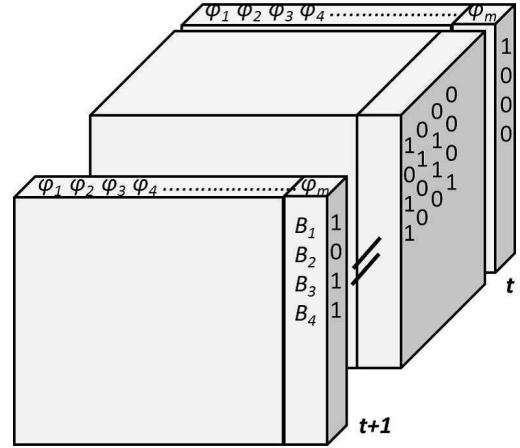

Fig. 2.   Multiple Propositions in Informing Action

V. MULTIPLE INFORMATION TENSOR

Thus far, we have restricted our concern to single information passing. However, we can represent the message passing of multiple information $\varphi_1, \varphi_2, \varphi_3, \cdots, \varphi_m$ as a tensor in Fig. 2.

The flat matrix in the front in Fig. 2 represents the resultant state of informing action. The i-th column is the belief states of the agents as to φ_i and the j-th row is what agent j believes, at time $t+1$. In the similar way, the flat matrix in the behind is that of belief states at time t. The in-between $n \times n \times m$-cuboid is a simultaneous communication, where n is the number of agents and m is the number of information. In order to clarify the relation of elements, we place the contravariant elements as superscripts and the covariant ones as subscripts as a tensor:

$$(B_j^\varphi)^{t+1} = (c_j^{i,\varphi})(B_i^\varphi)^t.$$

In Fig. 2, we only have shown the atomic propositions. We can add such composite propositions as $\varphi_1 \vee \varphi_2$ and $\varphi_1 \wedge \varphi_2$ simply in the figure, as these truth values are composed by φ_i's.

VI. MODEL UPDATING

A kripke frame for multiple agents is such $\mathfrak{M} = \langle \mathcal{W}, \mathcal{A}, \mathcal{R}_1, \cdots, \mathcal{R}_n, \mathcal{V} \rangle$ that $\mathcal{A}$ is a set of agents and $\mathcal{R}_i$ is the accessibility of belief modal operator B_i.

A belief state of an agent can be represented also in matrix, when we render (i, j)-element as the accessibility from possible world i to j of Kripke semantics [8]. For example,[3]

$$\begin{array}{c} \begin{array}{ccc} w_1 & w_2 & w_3 \end{array} \\ \begin{array}{c} w_1 \\ w_2 \\ w_3 \end{array} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \qquad (5)$$

represents $\mathcal{R} = \{w_1 R w_1 \ w_1 R w_2, \ w_2 R w_2, \ w_2 R w_3, \ w_3 R w_3\}$ in $\mathcal{W} = \{w_1, w_2, w_3\}$. In this matrix representation, the configuration of truth values directly shows the axioms of modality.

[3]We employ square brackets for the accessibility to distinguish it from the communication matrix.

For example, if the matrix is symmetric, it satisfies axiom of symmetricity (Axiom B). If the diagonal elements are all 1's, it is reflexive (Axiom T), If there is at least one 1 in each row, it satisfies seriality (Axiom D).

A belief change becomes a change in matrix. For example, let $\mathcal{V}(\varphi) = \{w_2, w_3\}$; then matrix (5) cannot satisfy $B_i^t\varphi$ as $\mathfrak{M}, w_1 \not\models B_i^t\varphi$ (for $w_1 R w_1$, $\mathfrak{M}, w_1 \not\models \varphi$). Here, we consider DEL style belief update (2), that is, to cut some of accessibility for an agent to come to believe an informed proposition; namely some 1's in the accessibility matrix at time t are replaced by 0's at $t + 1$. In the case of (5), if $(1, 1)$-element becomes 0, then $B_i^{t+1}\varphi$.

Since the accessibility with the valuation maps a belief state of a given agent to a truth value:

$$(\mathcal{R}_i, \mathcal{V}): B_i^t\varphi \to 1/0,$$

we identify such accessibility matrices with truth values in the following example.

Example 5: Suppose $\mathcal{V}(\varphi) = \{w_2, w_3\}$. A belief vector at time t is:[4]

$$B^t\varphi = (\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \cdots).$$

After a public announcement of φ from some agent, the revised belief vector becomes:

$$B^{t+1}\varphi = (\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \cdots). \blacksquare$$

Now, the multiple belief states of agents at time t becomes $(n \times m \times k^2)$-hypercuboid, where n is the number of agents, m is the number of information, and k is the number of possible worlds.[5]

VII. DISCUSSION

In the belief revision of multiple agents, as there are so many indices appears, we have represented them in linear logic. As we have shown, the collective belief state and the communication matrix include more then three indices, those matrices become hypercuboids.

We have shown that the consecutive informing action can be realized by a product of matrices, including the n-th power, with associativity and anti-communtativity. This discussion naturally leads us to the inversion of matrix, as

$$B^t = (c_{ij})^{-1} B^{t+1},$$

being the inversion regarded as belief contraction [5]. However, some communication matrices do not have their inversions, as their rank fails to be n. Furthermore, we cannot give proper semantics for value '-1', or '0' may appear on the diagonal elements in the inversion matrix. We need

to recognize that how we can *redo* the revision is difficult problem, especially in Kripke semantics.

In this paper, we evaluated formulae in the same possible world even though time shifts. We are now to intend that we regard the world itself as a temporal state, as:

$$\mathfrak{M}, t \models [c_{ij}^\varphi]\psi \iff \mathfrak{M}^{c_{ij}^\varphi}, t + 1 \models \psi.$$

Now, let us get back to the two issues: belief revision and nested modalities. We could not implement simultaneous arrivals of two different information to one agent, i.e.,

$$B_j^{t+1} := B_j^t \dotplus \{\varphi\} \dotplus \{\psi\},$$

regarding B_j^t as a belief set and '$\dotplus$' as revision operator. Note that how we can revise the belief of each agent is an independent topic from our formalism in this paper, and depends on the preference of revision. Also, it is difficult to send such propositions including modality as $B_j\varphi_2$, which results in the nested belief state. In addition, sending a negative formula also affects how we can revise the accessibility; these would be our common research topics in the community of belief update logic in future.

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[4] We show belief vectors in row vectors here just for visibility, although they are intrinsically column vectors.

[5] If we fix the number of atomic propositions to m, then the number of maximal possible worlds becomes $k = 2^m$.