Fuzziness in Partial Approximation Framework

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Abstract—In partial approximation spaces with Pawlakian approximation pairs, three partial membership functions are generated. These fuzzy functions rely on the lower and upper approximations of a set. They provide special type of fuzziness on the universe: all of them are partial functions and derived from the observed data relatively to available knowledge about the objects of the universe. With the help of these functions, three new approximation pairs are generated and so new approximation spaces appear effectively. Using not Pawlakian approximation pairs gives a special insight into the nature of general set approximations, and so new models of necessity and possibility can be given.

I. INTRODUCTION

Set approximations were invented by Pawlak in the early 1980’s which is known as rough set theory [1], [2], [3]. Its general scheme may be outlined as follows. Let a beforehand predefined family of subsets of the universe of objects be given. It is called the base system from which definable sets may be derived. Next, so-called lower and upper approximations can be formed with the help of definable sets via beforehand fixed rules in order to approximate any sets in the universe.

The starting point of rough set theory is a nonempty finite set $U$ of objects and an equivalence relation $\varepsilon$ on $U$ [3]. The equivalence classes are called $\varepsilon$-elementary sets.

Definable sets are any unions of $\varepsilon$-elementary sets. Any set $S \subseteq U$ can be naturally approximated by the lower and upper $\varepsilon$-approximations of $S$ which are denoted by $\varepsilon$ and $\varepsilon$, respectively. The former is the union of all $\varepsilon$-elementary sets which are the subsets of $S$, whereas the latter is the union of all $\varepsilon$-elementary sets which have a nonempty intersection with $S$.

A number of studies deal with the relationship between rough set theory and fuzzy set theory [4], [5], [6], [7], [8]. A detailed discussion of their connections and differences can be found, e.g., in [9], [10], [11].

There are many possibilities to establish a relationship between them [12], [13], [14], [15]. Just until now it has been generally accepted that the two theories are related but distinct and complementary to each other. Recently, however, Chakraborty has proposed a common ground relying on the classical rough membership function [16].

The classical rough membership function quantifies the degree of the relative overlap between a set $S \subseteq U$ and an $\varepsilon$-elementary set [10].\(^1\) As usual, it is defined by

$$\mu_\varepsilon^S(u) = \frac{|[u]_\varepsilon \cap S|}{|[u]_\varepsilon|},$$

where $| \cdot |$ is the cardinality of a set, and $[u]_\varepsilon$ denotes the $\varepsilon$-elementary set to which a $u \in U$ belongs.

Hence, we just obtain a fuzzy membership function $\mu_\varepsilon^S : U \rightarrow [0, 1]$ with

$$\mu_\varepsilon^S(u) = 1 \text{ if and only if } [u]_\varepsilon \subseteq S;$$
$$\mu_\varepsilon^S(u) > 0 \text{ if and only if } [u]_\varepsilon \cap S \neq \emptyset;$$
$$\mu_\varepsilon^S(u) = 0 \text{ if and only if } [u]_\varepsilon \cap S = \emptyset.$$

Thus, the rough membership function can be seen as a fuzzyfication of rough approximation, and $\mu_\varepsilon^S$ is a fuzzy subset of $U$ induced by $S$.

One of the main features of $\mu_\varepsilon^S$ is that it relies on the system of base sets, the system of equivalence classes. In other words, rough membership functions are generated by our knowledge (appearing, e.g., in an information system). This is a distinctive feature of rough membership functions in contrast with fuzzy membership functions [18]. Furthermore, following from the definition of $\mu_\varepsilon^S$, there are many constraints on the values of rough membership functions [12], [20], [21].

An important observation is that the Pawlakian lower and upper approximation pair can be reconstructed by employing the rough membership function. The well-known formulae are the following:

$$\varepsilon(S) = \{ u \in U | \mu_\varepsilon^S(u) = 1 \},$$
$$\varepsilon(S) = \{ u \in U | \mu_\varepsilon^S(u) > 0 \}.$$

In the terminology of fuzzy set theory, the lower and upper approximations $\varepsilon$ and $\varepsilon$ are the core and the support of the fuzzy set $\mu_\varepsilon^S$, respectively.

Nevertheless, Pawlakian set approximation has some very strong theoretical requirements:

- the system of base sets are total, i.e., their union gives back the universe;
- base sets are pairwise disjoint.

\(^1\)Note that the notion of a classical rough membership function was explicitly introduced by Pawlak and Skowron in [10]. Nevertheless, it has been used and studied earlier by many authors. For more historical remarks, see [17]. Moreover, such a coefficient has already been considered by Łukasiewicz in 1913 [18], [19].
In many cases, however, our knowledge does not fulfill these requirements:

- The partition shows the limit of our knowledge about the objects of the universe in the sense that two objects are indistinguishable if they belong to the same base set. On the other hand, it makes explicit our knowledge because we do distinguish two objects belonging to different base sets. Giving up the requirement of the pairwise disjoint property, the so-called covering-based rough set theory is obtained [22], [23], [24], [25], [26], [27].
- The universe may involve some objects without any information, i.e., base sets are not total. For instance, information systems often contain NULL values. In the papers [28], [29], the authors give a very general system of the set approximation giving up both the pairwise disjoint property and the covering of the universe. It is called the (general) partial approximation framework.

In this paper, the above procedure is transferred to a partial set approximation context:

1) First, in a partial approximation space with a Pawlakian approximation pair, three partial membership functions are defined in the style of the classical rough membership function.

2) Then, three approximation pairs are generated with the help of partial membership functions. The question is whether these approximation pairs meet (at least) the minimum requirements of approximation pairs, i.e., these pairs actually form approximation pairs in partial approximation spaces.

The rest of the paper consists of three parts. In Section 2, the basic notions and notations of partial approximation spaces are summarized. In Section 3, three approximation pairs are generated as outlined above, and it is shown that they meet the minimum requirements prescribed for approximation pairs in partial approximation spaces. Section 4 consist of some remarks on the logical application of partial membership functions.

II. PARTIAL APPROXIMATION OF SETS

A. Basic notions and notations

Let \( U \) be a nonempty finite set and \( \mathcal{B} \subseteq 2^U \) be a nonempty family of nonempty subsets of \( U \). \( U \) is the universe of objects, \( \mathcal{B} \) is the base system and its members are \( \mathcal{B} \)-sets or base sets [30], [29], [31], [32], [33].

If \( B \in \mathcal{B} \) is a union of a family of sets \( \mathcal{B} \subseteq \mathcal{B} \setminus \{B\} \), \( B \) is called reducible in \( \mathcal{B} \), otherwise \( B \) is irreducible in \( \mathcal{B} \).

A base system \( \mathcal{B} \) is single-layered if every base set is irreducible, and one-layered if the base sets are pairwise disjoint. Of course, a one-layered base system is single-layered. From any base systems, single-layered and one-layered base systems can be constructed [31].

By formulae, a base system \( \mathcal{B} \) is single-layered, if

\[
\forall B \in \mathcal{B} \ \forall B' \subseteq \mathcal{B} \setminus \{B\} \ (B \cap \bigcup B' \neq B).
\]

and one-layered, if

\[
\forall B \in \mathcal{B} \ \forall B' \subseteq \mathcal{B} \setminus \{B\} \ (B \cap \bigcup B' = \emptyset).
\]

Informally, a base system \( \mathcal{B} \) is single-layered if every nonempty union of base sets has at least one member which belongs to exactly one base set, whereas \( \mathcal{B} \) is one-layered if all members of every nonempty union of base sets belong to exactly one base set.

During the approximation process, a family of sets \( \mathcal{D}_\mathcal{B} \subseteq 2^U \) are applied. In the most general case, it is supposed only just that

1) \( \mathcal{D}_\mathcal{B} \) is an extension of \( \mathcal{B} \), i.e., \( \mathcal{B} \subseteq \mathcal{D}_\mathcal{B} \);
2) \( \emptyset \in \mathcal{D}_\mathcal{B} \).

Let \( l, u : 2^U \to 2^U \) be an ordered pair of mappings and denoted it by \( \langle l,u \rangle \).

The intended meaning of \( l \) and \( u \) is to express the lower and upper approximations of any subsets of \( U \). Hence, it is called an approximation pair. The next definition specifies its minimum requirements.

**Definition 1.** An approximation pair \( \langle l,u \rangle \) is a weak approximation pair if

(\( C_0 \)) \( \langle l(2^U), u(2^U) \rangle \subseteq \mathcal{D}_\mathcal{B} \) (definability of \( l \) and \( u \));

(\( C_1 \)) \( l \) and \( u \) are monotone, i.e. for all \( S_1, S_2 \in 2^U \) if \( S_1 \subseteq S_2 \) then \( l(S_1) \subseteq l(S_2) \) and \( u(S_1) \subseteq u(S_2) \) (monotonicity of \( l \) and \( u \));

(\( C_2 \)) \( u(\emptyset) = \emptyset \) (normality of \( u \));

(\( C_3 \)) if \( S \subseteq U \), then \( l(S) \subseteq u(S) \) (weak approximation property).

Clearly, \( l \) and \( u \) are many-to-one and \( u(2^U) \neq l(2^U) \subseteq \mathcal{D}_\mathcal{B} \) in general.

Informally, definable sets represent our available knowledge about the the objects of the universe. They can be thought of as tools, in more detail, base sets as primary tools and definable sets as derived tools. An approximation pair prescribes the utilization of tools in approximation processes.

It is reasonable that base sets as primary tools are exactly approximated from “lower side”. In classical rough set theory, however, definable sets are exactly approximated from “lower side” as well.

**Definition 2.** A weak approximation pair \( \langle l,u \rangle \) is

(\( C_4 \)) granular if \( B \in \mathcal{B} \),

then \( l(B) = B \) (\( l \) is granular),

(\( C_5 \)) standard if \( D \in \mathcal{D}_\mathcal{B} \), then \( l(D) = D \) (\( l \) is standard).

Of course, if \( l \) is standard, the granularity of \( l \) also holds.

The following proposition summarizes some simple consequences of the minimum requirements (\( C_0 \)–\( C_3 \)) in Definition 1 and the conditions (\( C_4 \)–\( C_5 \)) in Definition 2.

**Proposition 1.** Let \( \langle l,u \rangle \) be a weak approximation pair on \( U \).

1) \( l(\emptyset) = \emptyset \) (normality of \( l \)).

2) \( l \) is idempotent, i.e., \( l(l(S)) = l(S) \) for all \( S \in 2^U \), and \( l(2^U) = \mathcal{D}_\mathcal{B} \) if and only if \( l \) is standard.
3) a) If \( l(S) = S \), then \( S \in \mathcal{D}_B \).
   
   b) Let \( l \) be standard. Then, \( l(S) = S \) if and only if \( S \in \mathcal{D}_B \).

4) a) If \( l(U) = \bigcup \mathcal{D}_B \), then \( \bigcup \mathcal{D}_B \in \mathcal{D}_B \).
   
   b) Let \( l \) be standard. Then, \( l(U) = \bigcup \mathcal{D}_B \) if and only if \( \bigcup \mathcal{D}_B \in \mathcal{D}_B \).

The next definition deals with the question how lower and upper approximations relate to the approximated sets.

**Definition 3.** A weak approximation pair \((l, u)\) is

(C6) lower semi-strong if \( l(S) \subseteq S \) for all \( S \in 2^U \) (i.e., \( l \) is contractive);

(C7) upper semi-strong if \( S \subseteq u(S) \) for all \( S \in 2^U \) (i.e., \( u \) is extensive);

(C8) strong if it is lower and upper semi-strong, i.e., each subset \( S \in 2^U \) is bounded by \( l(S) \) and \( u(S) \): \( l(S) \subseteq S \subseteq u(S) \).

**Proposition 2.**

1) If \((l, u)\) is an upper semi-strong approximation pair on \( U \), then \( u(U) = U \) (co-normality of \( u \)).

2) If \((l, u)\) is an upper semi-strong approximation pair on \( U \) and \( l \) is standard, then \( l(U) = U \) (co-normality of \( l \)).

Based on the foregoing, a general set-theoretic partial approximation framework can be defined as follows.

**Definition 4.** The ordered 5-tuple \( \text{GAS}(U) = (U, \mathcal{B}, \mathcal{D}_B, l, u) \) whose components are defined as before, is called a (general) approximation space.

**Definition 5.** \( \text{GAS}(U) \) is a (general) total approximation space or simply total, if \( \mathcal{B} \) covers the universe, i.e., \( \bigcup \mathcal{B} = U \); otherwise \( \text{GAS}(U) \) is a (general) partial approximation space or simply partial.

**Definition 6.** \( \text{GAS}(U) \) relies on Pawlakian base, if \( \mathcal{B} \) is a partition of \( U \).

**Corollary 1.** \( \text{GAS}(U) \) relies on Pawlakian base if and only if its base system is total and one-layered.

**Definition 7.** The general approximation space \( \text{GAS}(U) \) is a weak/standard/lower semi-strong/upper semi-strong/strong approximation space, if the approximation pair \((l, u)\) is weak/standard/lower semi-strong/upper semi-strong/strong, respectively.

**B. Exactness in general approximation spaces**

In classical rough set theory, the notions of “crisp” and “definable” are inherently one and the same. In general approximation spaces, however, they can be differentiated.

**Definition 8.** Let \( \text{GAS}(U) \) be a weak approximation space and \( S \subseteq U \).

- \( S \) is crisp, if \( l(S) = u(S) \), otherwise \( S \) is rough.

If a set is crisp, its lower and upper approximations coincide with the approximated set only in strong approximation spaces. Furthermore, a crisp set is necessarily definable only in strong approximation spaces as well. However, it can easily be shown that a definable set is not necessarily crisp even in strong approximation spaces (133, Example 8). Consequently, in general approximations spaces, the notions of “crisp” and “definability” are generally not synonymous to each other.

**C. General approximation spaces with Pawlakian approximation pairs**

**Definition 9.** \( \text{GAS}(U) = (U, \mathcal{B}, \mathcal{D}_B, l, u) \) is a approximation space with a Pawlakian approximation pair, if

1) \( U \) is a finite nonempty set;

2) \( \mathcal{D}_B \) is strict finite union type, i.e., it is given by the following inductive definition:

   a) \( \emptyset \in \mathcal{D}_B \);

   b) \( \mathcal{B} \subseteq \mathcal{D}_B \);

   c) if \( B_1, B_2 \in \mathcal{B} \), then \( B_1 \cup B_2 \in \mathcal{D}_B \);

3) \((l, u)\) is a Pawlakian approximation pair, i.e.,

   a) \( l(S) = \bigcup U_S \), where \( U_S = \{ B \in \mathcal{B} \mid B \subseteq S \} \);

   b) \( u(S) = \bigcup U_S \), where \( U_S = \{ B \in \mathcal{B} \mid B \cap S \neq \emptyset \} \).

**Proposition 3.** Let \( \text{GAS}(U) \) be an approximation space with a Pawlakian approximation pair.

1) \( \text{GAS}(U) \) is a standard lower semi-strong approximation space.

2) \( \text{GAS}(U) \) is an upper semi-strong approximation space if and only if \( \mathcal{B} \) covers the universe.

**Definition 10.** Let \( \text{GAS}(U) \) be an approximation space with a Pawlakian approximation pair and \( S \subseteq U \). Then

\[
b(S) = \bigcup (U_S \setminus L(S))
\]

is called the boundary of \( S \).

Clearly, \( b(S) \subseteq u(S) \) for all \( S \subseteq U \).

**Corollary 2.** Let \( \text{GAS}(U) \) be an approximation space with a Pawlakian approximation pair.

1) In general, \( u(S) \setminus l(S) \subseteq b(S) \) for any \( S \subseteq U \).

2) If \( S \subseteq U \),

\[
b(S) = u(S) \setminus l(S) \Leftrightarrow b(S) \cap l(S) = \emptyset.
\]

**Proof:**

1) \( u \in u(S) \setminus l(S) \)

\[
\Rightarrow u \in \bigcup U_S \setminus u \notin \bigcup L(S)
\]

\[
\Rightarrow \exists B \in \mathcal{B} \ (u \in B \land B \in U_S \setminus B \notin L(S))
\]

\[
\Rightarrow u \notin \bigcup (U(S) \setminus L(S)) = b(S)
\]

2) \( \Rightarrow b(S) \cap l(S) = (u(S) \setminus l(S)) \cap l(S) = \emptyset. \)

\( \Leftarrow b(S) \)
\[ \begin{align*}
\approx & \ (b(S) \cap \#(S)) \cup (b(S) \cap (l(S))^c) \\
= & \ b(S) \cap (l(S))^c \\
\subseteq & \ u(S) \cap (l(S))^c = u(S) \setminus l(S),
\end{align*} \]

which are compared to (1), we get

\[ b(S) = u(S) \setminus l(S). \]

\[ \text{III. FUZZINESS IN PARTIAL APPROXIMATION SPACES WITH Pawlakian APPROXIMATION PAIRS} \]

Let \( \text{GAS}(U) = \langle U, \mathcal{B}, \mathcal{D}, l, u \rangle \) be a partial approximation space with a Pawlakian approximation pair. In other words, \( \text{GAS}(U) \) is an approximation space with a Pawlakian approximation pair and \( \bigcup \mathcal{B} \subseteq U \).

\[ \text{A. Partial membership functions} \]

If \( u \in U \), let \( \mathcal{N}_\mathcal{B}(u) = \{ B \in \mathcal{B} \mid u \in B \} \). The family of sets \( \mathcal{N}_\mathcal{B}(u) \) may be called the (reflexive) neighborhood system of \( u \) with respect to the base system \( \mathcal{B} \) [34], and its members are called the neighborhoods of \( u \).

Three different partial membership functions are defined in \( \text{GAS}(U) \) as follows [32], [33], [36], [38], [20].

\[ \text{Definition 11. Let } \text{GAS}(U) = \langle U, \mathcal{B}, \mathcal{D}, l, u \rangle \text{ be a partial approximation space with a Pawlakian approximation pair and } S \subseteq U. \]

\[ \mu_{p}^{b}, \mu_{f}^{b}, \mu_{p}^{p} : U \rightarrow [0,1] \text{ are optimistic/average/pessimistic partial membership functions, respectively, if} \]

\[ \mu_{p}^{b}(u) = \begin{cases} 
\max \left\{ \frac{|B \cap S|}{|B|} \mid B \in \mathcal{N}_\mathcal{B}(u) \right\}, & \text{if } u \in \mathcal{B}; \\
\text{undefined}, & \text{otherwise}; \\
\mu_{f}^{b}(u) = \begin{cases} 
\text{avg} \left\{ \frac{|B \cap S|}{|B|} \mid B \in \mathcal{N}_\mathcal{B}(u) \right\}, & \text{if } u \in \mathcal{B}; \\
\text{undefined}, & \text{otherwise}; \\
\mu_{p}^{p}(u) = \begin{cases} 
\min \left\{ \frac{|B \cap S|}{|B|} \mid B \in \mathcal{N}_\mathcal{B}(u) \right\}, & \text{if } u \in \mathcal{B}; \\
\text{undefined}, & \text{otherwise}. \\
\end{cases} 
\end{cases} \]

Remark 1. For the sake of brevity, we will use the symbol “\( \approx \)” in order to denote a member of \( \{ 0, a, p \} \).

In Definition 11, each partial membership function \( \mu_{f}^{b} \) forms a special type of fuzziness on \( U \) which is induced by the base system \( \mathcal{B} \), i.e., our available knowledge (primary tools) about the objects of the universe.

An important feature of each \( \mu_{f}^{b} \) is that it is a partial function.

Clearly, if \( \bigcup \mathcal{B} \subseteq U \), \( \mu_{f}^{b} \) is undefined for all \( u \in U \setminus \bigcup \mathcal{B} \). In other words, \( \text{dom } \mu_{f}^{b} = \bigcup \mathcal{B} \subseteq U \).

The following statements can easily be checked.

\[ \text{Proposition 4. Let } \text{GAS}(U) = \langle U, \mathcal{B}, \mathcal{D}, l, u \rangle \text{ be a partial approximation space with a Pawlakian approximation pair. Then, for any } S \subseteq U \text{ and } u \in U \]

1) \( \mu_{f}^{b}(u) = 1 \text{ if and only if} \)

\[ \exists B \in \mathcal{N}_\mathcal{B}(u) \ (B \subseteq S) \ (i.e., \mathcal{N}_\mathcal{B}(u) \cap \mathcal{L}(S) \neq \emptyset); \]

2) \( \mu_{f}^{b}(u) = 1, \mu_{p}^{b}(u) = 1 \text{ if and only if} \)

\[ \forall B \in \mathcal{N}_\mathcal{B}(u) \ (B \subseteq S) \ (i.e., \mathcal{N}_\mathcal{B}(u) \cap \mathcal{L}(S) \neq \emptyset); \]

3) \( \mu_{p}^{b}(u) > 0, \mu_{p}^{b}(u) > 0 \text{ if and only if} \)

\[ \exists B \in \mathcal{N}_\mathcal{B}(u) \ (B \cap S \neq \emptyset) \ (i.e., \mathcal{N}_\mathcal{B}(u) \cap \mathcal{U}(S) \neq \emptyset); \]

4) \( \mu_{p}^{b}(u) > 0 \text{ if and only if} \)

\[ \forall B \in \mathcal{N}_\mathcal{B}(u) \ (B \cap S \neq \emptyset) \ (i.e., \mathcal{N}_\mathcal{B}(u) \subseteq \mathcal{U}(S)); \]

5) \( \mu_{p}^{b}(u), \mu_{p}^{b}(u) = 0 \text{ if and only if} \)

\[ \forall B \in \mathcal{N}_\mathcal{B}(u) \ (B \cap S = \emptyset) \ (i.e., \mathcal{N}_\mathcal{B}(u) \cap \mathcal{S}(S) = \emptyset). \]

6) \( \mu_{p}^{b}(u) = 0 \text{ if and only if} \)

\[ \exists B \in \mathcal{N}_\mathcal{B}(u) \ (B \cap S = \emptyset). \]

Proposition 4 implies the following statements.

\[ \text{Corollary 3. Let } \text{GAS}(U) \text{ be a partial approximation space with a Pawlakian approximation pair. Then, for the optimistic partial membership function } \mu_{p}^{b}, \]

1) \( \mu_{p}^{b}(u) = 1 \text{ if and only if} \)

\[ u \in \mathcal{L}(S) \]

2) \( \mu_{p}^{b}(u) > 0 \text{ if and only if} \)

\[ u \in \mathcal{U}(S) \]

3) \( \mu_{p}^{b}(u) < 1 \text{ if and only if} \)

\[ u \notin \mathcal{L}(S) \]

4) \( \mu_{p}^{b}(u) = 0 \text{ if and only if} \)

\[ u \notin \mathcal{U}(S) \]

for any \( S \subseteq U \) and \( u \in U \).

\[ \text{Corollary 4. Let } \text{GAS}(U) \text{ be a partial approximation space with a Pawlakian approximation pair. Then, for the average partial membership function } \mu_{f}^{b}, \]

1) \( \mu_{f}^{b}(u) = 1 \text{ then } u \in \mathcal{L}(S) \]

2) \( \mu_{f}^{b}(u) > 0 \text{ then } u \in \mathcal{U}(S) \]

3) \( \mu_{f}^{b}(u) < 1 \text{ then } u \notin \mathcal{L}(S) \]

4) \( \mu_{f}^{b}(u) = 0 \text{ if and only if} \)

\[ u \notin \mathcal{U}(S) \]

for any \( S \subseteq U \) and \( u \in U \).

The different notions of necessity and possibility can be found in the definitions of partial membership functions \( \mu_{f}^{b} \).

The values \( \mu_{f}^{b}(u) \ (u \in U) \) of the partial membership functions defined above informally mean the following.

The case of optimistic partial membership function:

1) \( \mu_{f}^{b}(u) = 1 \), i.e., \( u \) has at least one neighborhood inside \( S \), \( u \) can certainly be classified as belonging to \( S \) in an optimistic sense;
2) if $\mu_S^a(u) > 0$, i.e., $u$ has at least one neighborhood wholly or partly inside $S$, $u$ can possibly be classified as belonging to $S$ in an optimistic sense;
3) if $0 < \mu_S^a(u) < 1$, i.e., $u$ does not have any neighborhood inside $S$ but has at least one neighborhood partly inside and partly outside $S$, $u$ cannot be classified as either belonging to $S$ or does not belonging to $S$ in an optimistic sense.

The case of the average partial membership function:
1) if $\mu_S^a(u) = 1$, i.e., all neighborhoods of $u$ are inside $S$, $u$ can certainly be classified as belonging to $S$ in average approach;
2) if $\mu_S^a(u) > 0$, i.e., $u$ has at least one neighborhood wholly or partly inside $S$, $u$ can possibly be classified as belonging to $S$ in average approach;
3) if $0 < \mu_S^a(u) < 1$, i.e., $u$ has a neighborhood not inside $S$ and has at least one neighborhood wholly or partly inside $S$, $u$ cannot be classified as either belonging to $S$ or does not belonging to $S$ in average approach.

The case of pessimistic partial membership function:
1) if $\mu_S^a(u) = 1$, i.e., all neighborhoods of $u$ are inside $S$, $u$ can certainly be classified as belonging to $S$ in a pessimistic sense;
2) if $\mu_S^a(u) > 0$, i.e., all neighborhoods of $u$ are wholly or partly inside $S$, $u$ can possibly be classified as belonging to $S$ in a pessimistic sense;
3) if $0 < \mu_S^a(u) < 1$, i.e., $u$ has a neighborhood not inside $S$ and all neighborhoods of $u$ are wholly or partly inside $S$, $u$ cannot be classified as either belonging to $S$ or does not belonging to $S$ in a pessimistic sense.

Last, for all three partial membership functions,
$$\mu_S^a(u) = \text{undefined}$$
indicates that we do not have any information about $u$. Consequently, defining membership degree for $u$ should be meaningless with respect to our knowledge about the objects of the universe.

In classical rough set theory, lower and upper approximations and the boundary can be reconstructed setting out from the membership function. In a fuzzy context, the reconstruction can be carried out by means of core and support of membership functions in a standard way.

As usual, for the partial membership function $\mu_S^a$, the core and support are the following:
$$\text{core}(\mu_S^a) = \{ u \in U \mid \mu_S^a(u) = 1 \};$$
$$\text{support}(\mu_S^a) = \{ u \in U \mid \mu_S^a(u) > 0 \}.$$  

Now, $l^*, u^* : 2^U \rightarrow 2^U$ approximation pair may be defined as usual:
$$l^*(S) = \text{core}(\mu_S^a) = \{ u \in U \mid \mu_S^a(u) = 1 \},$$
$$u^*(S) = \text{support}(\mu_S^a) = \{ u \in U \mid \mu_S^a(u) > 0 \}.$$  

1) The case of optimistic partial membership functions: In the optimistic partial membership function $\mu_S^a$, the optimistic lower and upper approximation pair is the following:
$$l^o(S) = \text{core}(\mu_S^a) = \{ u \in U \mid \mu_S^a(u) = 1 \} = \{ u \in l(S) \mid \exists B \in \mathcal{N}_S(u) (B \subseteq S) \} = l(S)$$
by Corollary 3 (1), and
$$u^o(S) = \text{support}(\mu_S^a) = \{ u \in U \mid \mu_S^a(u) > 0 \} = \{ u \in u(S) \mid \exists B \in \mathcal{N}_S(u) (B \cap S \neq \emptyset) \} = u(S)$$
by Corollary 3 (2).

Informally, $l^o(S)$ is a collection of such $u \in U$ which has at least one neighborhood included in $S$, and $u^o(S) = l(S)$. $u^o(S)$ is a collection of such $u \in U$ which has at least one neighborhood having nonempty intersection with $S$, and $u^o(S) = u(S)$.

In other words, in the case of optimistic partial membership function $\mu_S^a$, we get back the Pawlakian approximation pair $(l, u)$. It implies that $(l^o, u^o)$ meets the minimum requirements (C0)–(C3) and the conditions (C4)–(C5).

2) The case of average partial membership functions: In the case of the average partial membership function $\mu_S^a$, the average lower and upper approximation pair is the following:
$$l^a(S) = \text{core}(\mu_S^a) = \{ u \in U \mid \mu_S^a(u) = 1 \} = \{ u \in U \mid \forall B \in \mathcal{N}_S(u) (B \subseteq S) \} \subseteq l(S)$$
by Corollary 4 (1), and
$$u^a(S) = \text{support}(\mu_S^a) = \{ u \in U \mid \mu_S^a(u) > 0 \} = \{ u \in u(S) \mid \exists B \in \mathcal{N}_S(u) (B \cap S \neq \emptyset) \} = u(S)$$
by Corollary 4 (2).

Informally, $l^a(S)$ is a collection of such a $u \in U$ whose all neighborhoods included in $S$, and $l^a(S) \subseteq l(S)$. $u^a(S)$ is a collection of such $u \in U$ which has at least one neighborhood having nonempty intersection with $S$, and $u^a(S) = u(S)$.

That is, in the case of average partial membership function $\mu_S^a$, we get back the upper Pawlakian approximation map, but the Pawlakian lower approximation map has already changed.

**Proposition 5.** GAS$(U) = (U, \mathcal{B}, \mathcal{D}_1^u, l^*, u^*)$ is a weak general approximation space provided that $\mathcal{D}_1 \setminus \mathcal{D}_2 \in \mathcal{D}_2^u$ ($\mathcal{D}_1, \mathcal{D}_2 \in \mathcal{D}_2$).

**Proof:**

(C0)–(C2) They are straightforward.

(C3) If $u \in l^a(S)$, then $\forall B \in \mathcal{N}_S(u) (B \subseteq S)$, and so $\exists B \in \mathcal{N}_S(u) (B \cap S \neq \emptyset)$, i.e., $u \in u^a(S)$.
3) The case of pessimistic partial membership functions:
In the case of the pessimistic partial membership function μ_p, the pessimistic lower and upper approximation pair is the following:

\[ \mathcal{P}(S) = \text{core}(\mu_p) = \{ u \in U | \mu_p(u) = 1 \} \]
\[ \subseteq \{ u \in U | \forall B \in \mathcal{N}_B(u) (B \subseteq S) \} \]
by Corollary 5 (1), and

\[ \mathcal{U}(S) = \text{support}(\mu_p) = \{ u \in U | \mu_p(u) > 0 \} \]
\[ \subseteq \{ u \in U | \forall B \in \mathcal{N}_B(u) (B \cap S \neq \emptyset) \} \]
by Corollary 5 (2).

Informally, \( \mathcal{P}(S) \) is a collection of such \( u \in U \) whose all neighborhoods included in \( S \), and \( \mathcal{P}(S) \subseteq \mathcal{U}(S) \). \( \mathcal{U}(S) \) is a collection of such \( u \in U \) whose all neighborhoods have nonempty intersection with \( S \), and \( \mathcal{U}(S) \subseteq \mathcal{U}(S) \).

In the case of pessimistic partial membership function \( \mu_p \), both lower and upper Pawlakian approximation maps have changed.

**Proposition 6.** GAS\(_U\) = \( \langle U, \mathcal{B}, \mathcal{D}_{\mathcal{B}}, \mathcal{P}, \mathcal{U} \rangle \) is a weak general approximation space provided that \( \mathcal{D}_1 \setminus \mathcal{D}_2 \in \mathcal{D}_{\mathcal{B}} \) \( (\mathcal{D}_1, \mathcal{D}_2 \in \mathcal{D}_{\mathcal{B}}) \).

**Proof:**
(C0)–(C2) They are straightforward.
(C3) If \( u \in \mathcal{P}(S) \), then \( \forall B \in \mathcal{N}_B(u) (B \subseteq S) \), and so \( \forall B \in \mathcal{N}_B(u) (B \cap S \neq \emptyset) \), i.e., \( u \in \mathcal{U}(S) \).

The next proposition deals with the conditions (C4)–(C5) of average and pessimistic approximation pairs.

**Proposition 7.** Let \( \langle U, \mathcal{B}, \mathcal{D}_{\mathcal{B}}, \mathcal{P}, \mathcal{U} \rangle \) and \( \langle U, \mathcal{B}, \mathcal{D}_{\mathcal{B}}, \mathcal{P}, \mathcal{U} \rangle \) be weak approximation spaces whose components are defined as above.

If the base system \( \mathcal{B} \) is one–layered, \( \mathcal{D}_{\mathcal{B}} = \mathcal{P}_{\mathcal{B}} = \mathcal{D}_{\mathcal{B}} \) and the weak approximation pairs \( \langle \mathcal{P}, \mathcal{U} \rangle \) and \( \langle \mathcal{P}, \mathcal{U} \rangle \) are standard, i.e., \( \mathcal{P}(D) = D \) and \( \mathcal{U}(D) = D \) for all \( D \in \mathcal{D}_{\mathcal{B}} \).

**Proof:**
Since \( l \) is standard, \( \mathcal{P}(D) \subseteq \mathcal{U}(D) = D \) for all \( D \in \mathcal{D}_{\mathcal{B}} \).

On the other hand, \( \mathcal{B} \) is one–layered, and so every definable set \( D \in \mathcal{D}_{\mathcal{B}} \) is a finite union of pairwise disjoint base sets, e.g., \( D = B_1 \cup \ldots \cup B_n \), where \( B_i \)’s are pairwise disjoint. Moreover, for every \( u \in D \) there exists exactly one \( i \in \{ 1, 2, \ldots, n \} \) in such a way that \( \mathcal{N}_B(u) = \{ B_i \} \).

Hence, we get for all \( D \in \mathcal{D}_{\mathcal{B}} \).

\[ \mathcal{P}(D) = \{ u \in U | \forall B \in \mathcal{N}_B(u) (B \subseteq D) \} \]
\[ \geq \{ u \in D | \forall B \in \mathcal{N}_B(u) (B \subseteq D) \} \]
\[ = \{ u \in B_1 \cup \ldots \cup B_n | \forall B \in \mathcal{N}_B(u) (B \subseteq D) \} \]
\[ = \{ u \in B_1 | \forall B \in \mathcal{N}_B(u) (B \subseteq D) \} \]
\[ \cup \ldots \cup \{ u \in B_n | \forall B \in \mathcal{N}_B(u) (B \subseteq D) \} \]
\[ = B_1 \cup \ldots \cup B_n = D. \]
Therefore, \( \mathcal{P}(D) = D \).

The standard property of \( \mathcal{P} \) can be proved similarly.

IV. SOME REMARKS ON THE LOGICAL APPLICATIONS

In the previous sections, first, three partial membership functions have been defined in partial approximation spaces with Pawlakian approximation pairs, then three approximation pairs have been generated with the help of them. It has been shown that, among others, they meet the minimum requirements prescribed for approximation pairs in partial approximation spaces, i.e., they actually form approximation pairs.

Optimistic, average and pessimistic partial membership functions have already been studied by the second author from the logical point of view in [38], [32]. It turned out that they are in connection with decision–theoretic rough set models (DTRS) which can be considered as the probabilistic extensions of algebraic rough set models [37].

Optimistic, average and pessimistic partial membership functions may serve as a bases of the semantics of a partial first–order logic. In the paper [35], the semantic system of a partial first–order logic with three different types of partial membership functions is presented. The proposed logical system gives an exact possibility to introduce different semantic notions of logical consequence relations which can be used in order to make clear the consequences of our decisions.

V. CONCLUSION AND FUTURE WORK

In this paper, having defined three partial membership functions, three approximation pairs have been generated in partial approximation spaces with Pawlakian approximation pairs. We have investigated how these pairs meet the requirements prescribed for approximation pairs in partial approximation spaces. As a result, in this way we have constructed two not Pawlakian approximation pairs.

In the future, it should be worth performing similar investigations in partial approximation spaces setting out from arbitrary approximation pairs, in particular, which have been obtained in this paper.

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