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Examples of Ramanujan and expander graphs for practical applications

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***Abstract*—Expander graphs are highly connected sparse finite graphs. The property of being an expander seems significant in many of these mathematical, computational and physical contexts. Even more, expanders are surprizingly applicably applicable in other computational aspects: in the theory of error corecting codes and the theory of pseudorandomness, which are used in probabilistic algorithms. In this article we present a method to obtain a new examples of families of expanders graphs and some examples of Ramanujan graphs which are the best expanders. We describe properties of obtained graphs in comparison to previously known results. Numerical computations of eigenvalues presented in this paper have been computed with MATLAB.**

I. Introduction

THERE are many different algorithms in everyday life where graphs are used. The development of information technology has allowed various representations of graphs in the memory of a computer. Graph based algorithms are used, in particular, in cryptography, coding theory, car navigation systems, sociology, mobile robotics and even in computer games. Graphs used for different purposes often must have some special properties.

One of the most interesting features of the new graphs is their expansion property. Expander graphs are highly connected sparse finite graphs. This property seems to be very significant. From a practical viewpoint, these graphs resolve an extremal problem in communication network theory. Second, they fuse diverse branches of pure mathematics: number theory, representation theory and algebraic geometry.

Expander graphs are used to efficient error reduction in probabilistic algorithms. A randomized algorithm uses a source of pseudorandom bits. During execution, it takes random choices depending on those random data. However, to collect a reasonable collection of random bits is not an easy task. Algorithms that use the random input to reduce the expected running time or memory usage have a chance of producing an incorrect result. Using expander walks allows to achieve the same error probability, with much fewer random bits. The exact form of the exponential decay in error using expander walks and its dependence on the spectral gap was found by Gillman [6].

Constructions of the best expander graphs with a given regularity and order is not easy and in many cases, it is an open problem. In this article we present a method to obtain a new examples of families of expanders graphs and some examples of Ramanujan graphs which are the best expanders. We describe properties of obtained graphs in comparison to previously known results.

Throughout this paper, only undirected simple graphs without loops or multiple edges are considered. A distance between vertices v_1 and v_2 in the graph is the length of minimal path from v_1 to v_2. A graph is connected if for arbitrary pair of vertices v_1, v_2 there is a path from v_1 to v_2. The length g of the shortest cycle in a graph is called a *girth*, [3]. Bipartite graph is a graph whose vertices set V can be divided into two disjoint subsets V_1 and V_2 such that every edge connects a vertex in V_1 to one in V_2. We refer to bipartite graph $\Gamma(V_1 \cup V_2, E)$ as biregular one if the number of neighbors for vertices from each partition sets are constants s and t (bidegrees). We call a graph regular in the case $s = t$.

By the theorem of Alon and Boppana, large enough members of an infinite family of d-regular graphs with constant d satisfy the inequality $\lambda \geq 2\sqrt{d-1} - o(1)$, where λ is the second largest eigenvalue in absolute value. Ramanujan graphs are d-regular graphs for which the inequality $\lambda \leq 2\sqrt{d-1}$ holds.

We say that a family of regular graphs of bounded degree q of increasing order n *has an expansion constant c*, $c > 0$ if for each subset A of the vertex set X, $|X| = n$ with $|A| \leq n/2$ the inequality $|\partial A| \geq c|A|$ holds. The expansion constant of the family of q-regular graphs can be estimated via upper limit $q - \lambda_n$, $n \to \infty$, where λ_n is the second largest eigenvalue of family representative of order n. It is clear that a family of Ramanujan graphs of bounded degree q has the best expansion constant.

The first explicit expander graph family was constructed by Gregory Margulis in the 1970's via studies of Cayley graphs of large girth [13].

A family of graphs G_n is a family of graphs of increasing girth if $g(G_n)$ goes to infinity with the growth of n.

The *family of graphs of large girth* is an infinite family of simple regular graphs Γ_i of degree k_i and order v_i such that

$$g(\Gamma_i) \geq \gamma \log_{k_i} v_i, \tag{1}$$

where c is an independent of i constant (see [1], [2]).

A sparse graph has a small number of edges in comparison to the number of vertices. A simple relationship describing the density of the graph $\Gamma(V, E)$ is

$$D = \frac{2|E|}{|V|(|V|-1)}, \tag{2}$$

where $|E|$ is the number of edges of graph Γ and $|V|$ is the number of vertices. The maximal density is $D = 1$ when a graph is complete and the minimal density is 0 (Coleman & Moré 1983).

One of the very important classes of small world bipartite graphs with additional geometric properties important in this context, is a class of regular generalized m-gon, i.e. regular tactical configurations of diameter m and girth $2m$. For each parameter m, a regular generalized m-gon has degree $q+1$ and order $2(1+q+...+q^{m-1})$, [15].

According to the famous Feit-Higman theorem the regular thick (i.e. degree ≥ 3) generalized m-gons exist only for $m = 3, 4$ and 6, [5]. Thus Generalized Pentagon does not exist, in particular. We have the following properties of generalized polygons:

- the incidence graph of a projective plane $PG(2, q)$ has $|V| = \nu(q+1, 6) = 2(1+q+q^2)$ and $g = 6$,
- the incidence graph of a generalized quadrangle $GQ(q, q)$ has $|V| = \nu(q+1, 8) = 2(1+q+q^2+q^3)$ and $g = 8$,
- the incidence graph of a generalized hexagon $GH(q, q)$ has $|V| = \nu(q+1, 10) = 2(1+q+q^2+q^3+q^4+q^5)$ and $g = 12$.

II. Construction of the families

Described below families of graphs $D(n, \mathbb{F}_q)$ and $W(n, \mathbb{F}_q)$ can be used to obtain the new construction of expander graphs or even Ramanujan graphs.

Let F_q, where q is prime power, be a finite field. $CD(n, q)$ (connected components of $D(n, \mathbb{F}_q)$) and $W(n, \mathbb{F}_q)$ are connected, regular, bipartite families of graphs.

Traditionally in graph theory one subset of vertices in bipartite graphs is denoted by $V_1 = P$ and called a set of points and another one $V_2 = L$ is called a set of lines. Let P and L be two copies of Cartesian power ${\mathbb{F}_q}^n$, where $n \geq 2$ is an integer. Brackets and parenthesis will allow the reader to distinguish points and lines. In this note we concentrate on finite bipartite graphs on the vertex set $P \cup L$, where P and L are two copies of ${\mathbb{F}_q}^n$. If $z \in \mathbb{F}_q^n$, then $(z) \in P$ and $[z] \in L$.

First, we introduce the bipartite graph $D(\mathbb{F}_q)$, [9], with the following points and lines, which are infinite dimensional vectors over $\mathbb{F}_q$ written in the following way

$$(p) = (p_{0,1}, p_{1,1}, p_{1,2}, p_{2,1}, p_{2,2}, p'_{2,2}, p_{2,3}, ..., p_{i,i}, p'_{i,i}, p_{i,i+1}, p_{i+1,1} ...),$$

$$[l] = [l_{1,0}, l_{1,1}, l_{1,2}, l_{2,1}, l_{2,2}, l'_{2,2}, l_{2,3}, ..., l_{i,i}, l'_{i,i}, l_{i,i+1}, l_{i+1,1} ...].$$

The point (p) is incident with the line $[l]$, which is written by the formula: $(p)I[l]$, if the following relations between their coordinates hold:

$$\begin{cases} l_{1,1} - p_{1,1} = l_{1,0} p_{0,1} \\ l_{1,2} - p_{1,2} = l_{1,1} p_{0,1} \\ l_{2,1} - p_{2,1} = l_{0,1} p_{1,1} \\ l_{i,i} - p_{i,i} = l_{0,1} p_{i-1,i} \\ l'_{i,i} - p'_{i,i} = l_{i,i-1} p_{0,1} \\ l_{i,i+1} - p_{i,i+1} = l_{i,i} p_{0,1} \\ l_{i+1,i} - p_{i+1,i} = l_{0,1} p'_{i,i} \end{cases} \tag{3}$$

where $i \geq 2$. The set of vertices of the graph $D(\mathbb{F}_q)$ of this infinite structure is $V = P \cup L$ and the set of edges consisting of all pairs $\{(p), [l]\}$ for which $(p)I[l]$.

For each positive integer $n > 2$ we obtain a finite incidence structure $(P_n, L_n, I_n)_D$ as follows. Firstly, P_n and L_n are obtained from P and L, respectively, by projecting each vector onto its n initial coordinates with respect to the natural order. The incidence I_n is then defined by imposing the first $n-1$ incidence equations and ignoring all others. The graph corresponding to the finite incidence structure (P_n, L_n, I_n) is denoted by $D(n, \mathbb{F}_q)$. $D(n, \mathbb{F}_q)$ becomes disconnected for $n \geq 6$. Graphs $D(n, \mathbb{F}_q)$ are edge transitive. It means that their connected components are isomorphic. A connected component of $D(n, \mathbb{F}_q)$ is denoted by $CD(n, \mathbb{F}_q)$. Notice that all connected components of infinite graph $D(\mathbb{F}_q)$ are q-regular trees.

The family of graphs $D(n, \mathbb{F}_q)$ is a family of q-regular, bipartite graphs of large girth (1). Graphs $D(n, \mathbb{F}_q)$, $n \geq 2$ of fixed degree q form a family of expanders with the second largest eigenvalue bounded from above by $2\sqrt{q}$, [9]. So, family $D(n, \mathbb{F}_q)$ consist of "almost Ramanujan graphs". A graph $D(n, \mathbb{F}_q)$ has practical application in the construction of error correcting codes. Firstly LDPC codes based on graphs $CD(n, \mathbb{F}_q)$ were described in [7]. They are still in practical use.

Let us consider an alternative way of presentation of q-regular infinite graph via equations over finite field F_q We consider an infinite graph $W(\mathbb{F}_q)$ with the points and lines:

$$(p) = (p_{0,1}, p_{1,1}, p_{1,2}, p_{1,3}, p_{1,4}, ..., p_{1,i}, ...),$$

$$[l] = [l_{1,0}, l_{1,1}, l_{1,2}, l_{1,3}, l_{1,4}, ..., l_{1,i}, ...].$$

$W(\mathbb{F}_q)$ is a graph of infinite incidence structure $(P, L, I)_W$ such that a point (p) is incident with the line $[l]$ $((p)I[l])$, if the following relations between their coordinates hold:

$$l_{1,i} - p_{1,i} = l_{1,i-1} p_{0,1} \tag{4}$$

Like in the case of $D(\mathbb{F}_q)$ for each positive integer $n > 2$ we obtain an finite incidence structure $(P_n, L_n, I_n)_W$ where P_n and L_n are obtained from P and L, respectively, by projecting each vector onto its n initial coordinates with respect to the natural order. The incidence I_n is then defined by imposing the first $n-1$ incidence equations and ignoring all others.

The graph corresponding to the finite incidence structure (P_n, L_n, I_n) is denoted by $W(n, \mathbb{F}_q)$.

The family $W(n, \mathbb{F}_q)$ is a family of q-regular, bipartite graphs with $g = 8$, given by a nonlinear system of equations.

By theorem 4.2 in [14] Wenger graph $W(n, \mathbb{F}_q)$ graph is an edge transitive one.

In fact, $W(n, \mathbb{F}_q)$ form a family of small world graphs. There is a conjecture that $CD(n, \mathbb{F}_q)$ is another family of small world graphs.

Firstly, let us consider an ordinary $n+1$-gon as a bipartite graph with vertex set $V = \{(1), (2), \ldots, (n+1)\} \cup \{[1,2], [2,3], \ldots, [n, n+1], [n+1, 1]\}$. We can write the incidence relation I in $n+1$-gon as follows:

$$(A)I[a, b] \Longleftrightarrow A = a \vee A = b.$$

A line is incident with point if this point belong to this line.

Graphs $G(n+1, \Gamma(n, \mathbb{F}_q))$ correspond to incidence structure with the point set P, the line set L and symmetric incidence relation I_G. Γ is a q-regular bipartite family of graphs defined by systems of equations. Then the number of vertices in graph G is $|V| = 2(1 + q + q^2 + \ldots + q^n)$. The graph is bipartite $V = P \cup L$ and a set V consists of:

2 elements of type t_0–$((1), \emptyset)$ and $[[1,2], \emptyset]$,

$2q$ elements of type t_1–$((2), *)$ and $[[1,2], *]$,

$2q^2$ elements of type t_2–$((n+1), *, *)$ and $[[2,3], *, *]$,

$\vdots$

$2q^n$ elements of type t_n–$((\lceil \frac{n+3}{2} \rceil), \underbrace{*, \ldots, *}_{n})$ and

$[[\lfloor \frac{n+3}{2} \rfloor, \lfloor \frac{n+5}{2} \rfloor], \underbrace{*, \ldots, *}_{n}]$.

Each $*$ represents an arbitrary element from $\mathbb{F}_q$. Brackets and parenthesis will allow the reader to distinguish points $(\cdot)$ and lines $[\cdot]$. The set of edges consisting of all pairs $\{(p), [l]\}$ for which $(p)I_G[l]$.

The incidence relation I_G in graphs $G(n+1, \Gamma(n, \mathbb{F}_q))$ is described as follows. A point of type t_0– $((1), \emptyset)$ is connected by an edge with a line of type t_0– $[[1,2], \emptyset]$ and lines of type t_1. A line of type t_0– $[[1,2], \emptyset]$ is connected by an edge with a point of type t_0– $((1), \emptyset)$ and points of type t_1. For $n \geq x, y \geq 1$, the point $(p) = ((A), \alpha_1, \alpha_2, \ldots, \alpha_x)$ of type t_x is incident $(p)I_G[l]$ with the line $[l] = [[a, b], \beta_1\beta_2, \ldots, \beta_y$ of type t_y if $A = a \vee A = b$ and the following hold:

$$\begin{cases} \alpha_1 = \beta_1, \alpha_2 = \beta_2, \ldots, \alpha_x = \beta_{y-1}, \text{ for } x + 1 = y \\ \alpha_1 = \beta_1, \alpha_2 = \beta_2, \ldots, \alpha_{x-1} = \beta_y, \text{ for } x = y + 1 \\ (\alpha_1, \alpha_2, \ldots, \alpha_n)I[\beta_1, \beta_2, \ldots, \beta_n] \text{ in } \Gamma, \text{ for } x = y = n \end{cases} \tag{5}$$

If we rewrite incidence relation for a graph $D(n, \mathbb{F}_q)$ with the notation for points as lines as for graph $W(n, \mathbb{F}_q)$:

$$(p) = (p_{0,1}, p_{1,1}, p_{1,2}, p_{1,3}, p_{1,4}, ..., p_{1,i}, ...),$$

$$[l] = [l_{1,0}, l_{1,1}, l_{1,2}, l_{1,3}, l_{1,4}, ..., l_{1,i}, ...],$$

($p_{2,1} = p_{1,3},\ p_{2,2} = p_{1,4},\ p'_{2,2} = p_{1,5}$ and $l_{2,1} = l_{1,3},\ l_{2,2} = l_{1,4},\ l'_{2,2} = l_{1,5}$) then the first 5 equations describing incidence relations for graph $D(n, \mathbb{F}_q)$ can be written as follows:

$$\begin{cases} l_{1,1} - p_{1,1} = l_{1,0}p_{0,1} \\ l_{1,2} - p_{1,2} = l_{1,1}p_{0,1} \\ l_{1,3} - p_{1,3} = l_{0,1}p_{1,1} \\ l_{1,4} - p_{1,4} = l_{0,1}p_{1,2} \\ l_{1,5} - p_{1,5} = l_{1,3}p_{0,1} \end{cases} \tag{6}$$

and tables II, III,IV, V describe incidence relations I_G for "small" representatives of the family.

TABLE I
REGULARITY AND ORDER FOR SOME REPRESENTATIVES OF THE FAMILY

Construction	Regularity	|V|
$G(3, D(2, \mathbb{F}_q)) \cong G(3, W(2, \mathbb{F}_q))$	$q+1$	$2(1+q+q^2)$
$G(4, D(3, \mathbb{F}_q)) \cong G(4, W(3, \mathbb{F}_q))$	$q+1$	$2(1+q+q^2+q^3)$
$G(5, D(4, \mathbb{F}_q))$	$q+1$	$2(1+q+q^2+q^3+q^4)$
$G(5, W(4, \mathbb{F}_q))$	$q+1$	$2(1+q+q^2+q^3+q^4)$
$G(6, W(5, \mathbb{F}_q))$	$q+1$	$2(1+q+q^2+q^3+q^4+q^5)$
$G(6, D(5, \mathbb{F}_q))$	$q+1$	$2(1+q+q^2+q^3+q^4+q^5)$

III. COMPARISON WITH PREVIOUSLY KNOWN RESULTS

The graphs $G(n+1, \Gamma(n, \mathbb{F}_q))$ have a structure which is some aspects similar to generalized polygons. They are $q+1$ regular graphs and have the same number of vertices for fixed $n+1 = 3, 4, 6$ as generalized polygons. According to the famous Feit-Higman theorem regular thick polygons exist only for $n+1 = 3, 4, 6$ (see [5]). For $n+1 = 2$ the described construction yields classical projective plane which is a generalized 3-gon and has the second eigenvalue $\lambda_1 = \sqrt{q}$. To show that the constructed graphs for $n+1 = 4, 6$ are not isomorphic to generalized quadrangles and hexagons we prove the following theorem.

Theorem 1. *Family of graphs $G(n+1, D(n, \mathbb{F}_q))$ and $G(n+1, W(n, \mathbb{F}_q))$ are families of graphs of girth 6.*

Proof. Graphs $G(n+1, D(n, \mathbb{F}_q))$ and $G(n+1, W(n, \mathbb{F}_q))$ are bipartite so there is no cycle C_3 and C_5. Because of the structure of this families there are two possibilities of appearence C_4:

1) There is a cycle C_4 consisting of two points of type t_n and two lines of type t_n. But it means that $D(n, \mathbb{F}_q)$ or $W(n, \mathbb{F}_q)$ have cycles of length 4. and we know from [14], [9] that $g(D(n, \mathbb{F}_q)) \geq 6$ and $g(W(n, \mathbb{F}_q)) \geq 6$.
2) There exists two vertices v_1 and v_2 of type t_n in the same branch which are separated by a path of length 2 ($[l_1]I(p_2)I[l_2]$), where p_2 is of type t_{n-1} and have a common neighbor of type t_n.

Suppose that the graph has C_4. v_1 and v_2 are from the same branch so they have equal coordinates except the last one. Assume without loss of generality that these are lines and denote them as follows:

$[l_1] = [[\lfloor \frac{n+3}{2} \rfloor, \lfloor \frac{n+5}{2} \rfloor], *_1, *_2, \ldots, *_{n-1}, Y_1]$,

TABLE II
INCIDENCE RELATIONS FOR GRAPH $G(3, W(2, \mathbb{F}_q)) \cong G(3, D(2, \mathbb{F}_q))$

	$((1), \emptyset)$	$((2), p_{0,1})$	$((3), p_{0,1}, p_{1,1})$
$[[1,2], \emptyset]$	$+$	$+$	$-$
$[[3,1], l_{1,0}]$	$+$	$-$	$+: p_{0,1} = l_{1,0}$
$[[2,3], l_{1,0}, l_{1,1}]$	$-$	$+: p_{0,1} = l_{1,0}$	$+: l_{1,1} - p_{1,1} = l_{1,0} p_{1,0}$ the first incydence equation for used graph

TABLE III
INCIDENCE RELATIONS FOR GRAPH $G(4, W(3, \mathbb{F}_q)) \cong G(4, D(3, \mathbb{F}_q))$

	$((1), \emptyset)$	$((2), p_{0,1})$	$((4), p_{0,1}, p_{1,1})$	$((3), p_{0,1}, p_{1,1}, p_{1,2})$
$[[1,2], \emptyset]$	$+$	$+$	$-$	$-$
$[[4,1], l_{1,0}]$	$+$	$-$	$+: p_{0,1} = l_{1,0}$	$-$
$[[2,3], l_{1,0}, l_{1,1}]$	$-$	$+: p_{0,1} = l_{1,0}$	$-$	$+: p_{0,1} = l_{1,0}$ $p_{1,1} = l_{1,1}$
$[[3,4], l_{1,0}, l_{1,1}, l_{1,2}]$	$-$	$-$	$+: p_{0,1} = l_{1,0},$ $p_{1,1} = l_{1,1}$	$+: l_{1,1} - p_{1,1} = l_{1,0} p_{0,1},$ $l_{1,2} - p_{1,2} = l_{1,1} p_{0,1}$

TABLE IV
INCIDENCE RELATIONS FOR GRAPH $G(5, W(4, \mathbb{F}_q))$ AND $G(5, D(4, \mathbb{F}_q))$

	$((1), \emptyset)$	$((2), p_{0,1})$	$((5), p_{0,1}, p_{1,1})$	$((3), p_{0,1}, p_{1,1}, p_{1,2})$	$((4), p_{0,1}, p_{1,1}, p_{1,2}, p_{1,3})$
$[[1,2], \emptyset]$	$+$	$+$	$-$	$-$	$-$
$[[1,5], l_{1,0}]$	$+$	$-$	$+: p_{0,1} = l_{1,0}$	$-$	$-$
$[[2,3], l_{1,0}, l_{1,1}]$	$-$	$+: p_{0,1} = l_{1,0}$	$-$	$+: p_{0,1} = l_{1,0}$ $p_{1,1} = l_{1,1}$	$-$
$[[4,5], l_{1,0}, l_{1,1}, l_{1,2}]$	$-$	$-$	$+: p_{0,1} = l_{1,0},$ $p_{1,1} = l_{1,1}$	$-$	$+: p_{0,1} = l_{1,0},$ $p_{1,1} = l_{1,1}$ $p_{1,2} = l_{1,2}$
$[[3,4], l_{1,0}, l_{1,1}, l_{1,2}, l_{1,3}]$	$-$	$-$	$-$ $-$	$+: p_{0,1} = l_{1,0},$ $p_{1,1} = l_{1,1}$ $p_{1,2} = l_{1,2}$	$+: l_{1,1} - p_{1,1} = l_{1,0} p_{0,1},$ $l_{1,2} - p_{1,2} = l_{1,1} p_{0,1}$ $l_{1,3} - p_{1,3} = l_{1,2} p_{0,1}$

TABLE V
INCIDENCE RELATIONS FOR GRAPH $G(6, W(5, \mathbb{F}_q))$ AND $G(6, D(5, \mathbb{F}_q))$

	$((1), \emptyset)$	$((2), p_{0,1})$	$((6), p_{0,1}, p_{1,1})$	$((3), p_{0,1}, p_{1,1}, p_{1,2})$	$((5), p_{0,1}, \ldots, p_{1,3})$	$((4), p_{0,1}, \ldots, p_{1,4})$
$[[1,2], \emptyset]$	$+$	$+$	$-$	$-$	$-$	$-$
$[[1,6], l_{1,0}]$	$+$	$-$	$+: p_{0,1} = l_{1,0}$	$-$	$-$	$-$
$[[2,3], l_{1,0}, l_{1,1}]$	$-$	$+: p_{0,1} = l_{1,0}$	$-$	$+: p_{0,1} = l_{1,0}$ $p_{1,1} = l_{1,1}$	$-$	$-$
$[[5,6], l_{1,0}, l_{1,1}, l_{1,2}]$	$-$	$-$	$+: p_{0,1} = l_{1,0},$ $p_{1,1} = l_{1,1}$	$-$	$+: p_{0,1} = l_{1,0},$ $p_{1,1} = l_{1,1}$ $p_{1,2} = l_{1,2}$	$-$
$[[3,4], l_{1,0}, \ldots, l_{1,3}]$	$-$	$-$	$-$	$+: p_{0,1} = l_{1,0},$ $p_{1,1} = l_{1,1}$ $p_{1,2} = l_{1,2}$	$-$	$+: p_{0,1} = l_{1,0},$ $p_{1,1} = l_{1,1}$ $p_{1,2} = l_{1,2}$ $p_{1,3} = l_{1,3}$
$[[4,5], l_{1,0}, \ldots, l_{1,4}]$	$-$	$-$	$-$	$-$	$+: p_{0,1} = l_{1,0},$ $p_{1,1} = l_{1,1}$ $p_{1,2} = l_{1,2}$ $p_{1,3} = l_{1,3}$	$+: l_{1,1} - p_{1,1} = l_{1,0} p_{0,1},$ $l_{1,2} - p_{1,2} = l_{1,1} p_{0,1}$ $l_{1,3} - p_{1,3} = l_{1,2} p_{0,1}$ $l_{1,4} - p_{1,4} = l_{1,3} p_{0,1}$

$[l_2] = [[\lfloor \frac{n+3}{2} \rfloor, \lfloor \frac{n+5}{2} \rfloor], *_1, *_2, \ldots, *_{n-1}, Y_2]$.
Denote their neighbor of type t_n as:
$(p_1) = ((\lceil \frac{n+3}{2} \rceil), \alpha_1, \alpha_2, \ldots, \alpha_n)$
and their neighbor (p_2) of type t_{n-1} from the same branch as:
$(p_2) = ((\lfloor \frac{n+3}{2} \rfloor), *_1, *_2, \ldots, *_{n-1})$. If in the graph $G(n+1, W(n, \mathbb{F}_q))$: $(p_1)I[l_1]$ and $(p_1)I[l_2]$ accordingly to (4) the following relations hold:

$$\begin{array}{cc}
*_2 - \alpha_2 = *_1\alpha_1 & *_2 - \alpha_2 = *_1\alpha_1 \\
*_3 - \alpha_3 = *_2\alpha_1 & *_3 - \alpha_3 = *_2\alpha_1 \\
*_4 - \alpha_4 = *_3\alpha_1 & *_4 - \alpha_4 = *_3\alpha_1 \\
\vdots & \vdots \\
*_{n-1} - \alpha_{n-1} = *_{n-2}\alpha_1 & *_{n-1} - \alpha_{n-1} = *_{n-2}\alpha_1 \\
Y_1 - \alpha_n = *_{n-1}\alpha_1 & Y_2 - \alpha_n = *_{n-1}\alpha_1
\end{array}$$

From the above equality one can see that Y_1 and Y_2 are uniquely determined by the remaining coordinates and $Y_1 = Y_2$. So we got a contradiction.

Analogous procedure can be performed for the graph $G(n+1, D(n, \mathbb{F}_q))$. Any graph $G(n+1, D(n, \mathbb{F}_q))$ and $G(n+1, W(n, \mathbb{F}_q))$ without vertices of type t_n is a tree and does not have any cycle.

For an arbitrary $n \geq 2$ in $G(n+1, D(n, \mathbb{F}_q))$ and $G(n+1, W(n, \mathbb{F}_q))$ there is a cycle of length 6:

$$[[\lfloor \tfrac{n+3}{2} \rfloor, \lfloor \tfrac{n+5}{2} \rfloor], \underbrace{0, 0, \ldots, 0}_{n}] I((\lfloor \tfrac{n+3}{2} \rfloor), \underbrace{0, 0, \ldots, 0}_{n-1})\ I$$
$$[[\lfloor \tfrac{n+3}{2} \rfloor, \lfloor \tfrac{n+5}{2} \rfloor], \underbrace{0, 0, \ldots, 0}_{n-1}, 1] I((\lceil \tfrac{n+3}{2} \rceil), \underbrace{0, 0, \ldots, 0}_{n-1}, 1)\ I$$
$$[[\lceil \tfrac{n+3}{2} \rceil, \lceil \tfrac{n+5}{2} \rceil], \underbrace{0, 0, \ldots, 0}_{n-1}] I((\lceil \tfrac{n+3}{2} \rceil), \underbrace{0, 0, \ldots, 0}_{n})\ I$$
$$[[\lfloor \tfrac{n+3}{2} \rfloor, \lfloor \tfrac{n+5}{2} \rfloor], \underbrace{0, 0, \ldots, 0}_{n}].$$

□

The above theorem leads to the following conclusion.

Corollary 2. *For $n \geq 3$ graphs $G(n+1, D(n, \mathbb{F}_q))$ and $G(n+1, W(n, \mathbb{F}_q))$ are not isomorphic to generalized polygons.*

IV. Expanding and other properties

The families $G(n+1, \Gamma(n, \mathbb{F}_q))$ consist of bipartite graphs with $|V| = 2(1 + q + q^2 + \ldots + q^n)$ vertices and $(q+1)(1 + q + q^2 + \ldots + q^n)$ edges. $G(n+1, D(n, \mathbb{F}_q))$ and $G(n+1, W(n, \mathbb{F}_q))$ are $q+1$-regular sparse graphs and the density according to (2) is

$$\frac{q+1}{2(q + \ldots + q^n) + 1}.$$

Fig. 1. shows the graph $G(3, \Gamma(2, \mathbb{F}_2))$ with 14 vertices $V = \{((1), \emptyset), ((2), 0), ((2), 1), ((3), 0, 0), ((3), 0, 1), ((3), 1, 0), ((3), 1, 1)\} \cup \{[[1, 2], \emptyset], [[1, 3], 0], [[1, 3], 1], [[2, 3], 0, 0], [[2, 3], 0, 1], [[2, 3], 1, 0], [[2, 3], 1, 1]\}$ and density $\frac{3}{13}$. The red vertices correspond to points and the blue vertices correspond to lines.

Each of the representatives of the presented family is $q+1$-regular graph so the first eigenvalue of the adjacency matrix, corresponding to this graph, is $\lambda_0 = q + 1$. Let us denote the second eigenvalue by $\lambda_1 = \max_{\lambda_i \neq q+1} |\lambda_i|$. On the basis of numerical calculations included in tables (VI), (VII), (IX), (X), (XI) we state the following conclusions:

- For $q = 2, 3, 4, 5, 7, 9, 11, 13, 17, 19, 23$ the constructed graphs $G(4, D(3, \mathbb{F}_q))$=$G(4, W(3, \mathbb{F}_q))$ are Ramanujan graphs. The spectral gap increases with the value of q. Basing on on this observation we have included Conjecture 1.
- For $q = 2, 3, 4, 5, 7, 11$ the constructed graphs $G(5, D(4, \mathbb{F}_q))$ and $G(5, W(4, \mathbb{F}_q))$ are Ramanujan graphs. The spectral gap $|\lambda_0 - \lambda_1| = |q + 1 - 2\sqrt{q}|$ increases with the value of q and basing on this observation we have included Conjecture 2.
- For $q = 2, 3, 4, 5, 7$ the constructed graphs $G(6, D(5, \mathbb{F}_q))$ and $G(6, W(5, \mathbb{F}_q))$ are expander graphs. The spectral gap for graph $G(6, W(5, \mathbb{F}_q))$ increases with the value of q and for graph $G(6, D(5, \mathbb{F}_p))$ increases with the value of p. This observation allows us to formulate Conjecture 3.

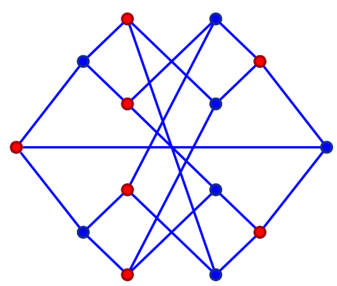

Fig. 1. $G(3, \Gamma(2, \mathbb{F}_2))$ with $|V| = 2(1 + 2 + 2^2) = 14$.

TABLE VI
Expanding properties of $G(4, D(3, \mathbb{F}_q))$=$G(4, W(3, \mathbb{F}_q))$

Number field	regularity $q+1$ first eigenvalue	second eigenvalue	$2\sqrt{q}$	$\|V\|$
$\mathbb{F}_2$	3	2.2882	2.8284	30
$\mathbb{F}_3$	4	2.8025	3.4641	80
$\mathbb{F}_4$	5	3.2361	4	170
$\mathbb{F}_5$	6	3.6180	4.4721	312
$\mathbb{F}_7$	8	4.2809	5.2915	800
$\mathbb{F}_{11}$	12	5.3664	6.6332	2928
$\mathbb{F}_{13}$	14	5.8339	7.2111	4760
$\mathbb{F}_{17}$	18	6.6713	8.2462	10440
$\mathbb{F}_{19}$	20	7.0528	8.7178	14480
$\mathbb{F}_{23}$	24	7.7598	9.5917	25440

We can use a finite ring $\mathbb{Z}_s$ and modulo operation instead of $\mathbb{F}_q$. The incidence relation for graph $G(n+1, \Gamma(n, \mathbb{Z}_s))$ can be described the same as for graph $G(n+1, D(n, \mathbb{F}_q))$. When we choose $s = 2r$ then the graphs $G(4, D(3, \mathbb{Z}_{2r}))$=$G(4, W(3, \mathbb{Z}_{2r}))$ have interesting constant value of spectral gap: $|\lambda_0 - \lambda_1| = 1$, for $2 \leq r \leq 13$. The results of such calculations (Tab. VII) allow us to formulate Conjecture 4. When we choose $s = 3r$, where $r = 3, 5, 7, 9$, then the graphs $G(4, D(3, \mathbb{Z}_{3r}))$=$G(4, W(3, \mathbb{Z}_{3r}))$ have an interesting value of the second largest eigenvalue

$$\lambda_1 = 2\sqrt{3r \left\lfloor \frac{r}{2} \right\rfloor + \frac{3r}{2}}$$

and the spectral gap increases with the value of q (Tab. VIII). Basing on on this observation we can not say whether for arbitrarily large s above formula is true. If yes, there is a question about r: if it should be a prime power ($\neq 2^l$) or an odd number? To answer this question we must calculate λ_1 for $r = 15$ but this case can not be investigated with MATLAB in computer with 8GB RAM. The adjacency matrix in this case has 186392×186392 elements.

TABLE VII
EXPANDING PROPERTIES OF $G(4, D(3, \mathbb{Z}_{2r}))$=$G(4, W(3, \mathbb{Z}_{2r}))$

Finite ring	regularity $q+1$ first eigenvalue	second eigenvalue	$2\sqrt{q}$	$\|V\|$
$\mathbb{Z}_4$	5	4	4	170
$\mathbb{Z}_6$	7	6	4.899	518
$\mathbb{Z}_8$	9	8	5.6569	1170
$\mathbb{Z}_{10}$	11	10	6.3246	2222
$\mathbb{Z}_{12}$	13	12	6.9282	3770
$\mathbb{Z}_{14}$	15	14	7.4833	5910
$\mathbb{Z}_{16}$	17	16	8	8738
$\mathbb{Z}_{18}$	19	18	8.4853	12350
$\mathbb{Z}_{20}$	21	20	8.9443	16842
$\mathbb{Z}_{22}$	23	22	9.3808	22310
$\mathbb{Z}_{24}$	25	24	9.798	28850
$\mathbb{Z}_{26}$	27	26	10.198	36558

TABLE VIII
EXPANDING PROPERTIES OF $G(4, D(3, \mathbb{Z}_{3r}))$=$G(4, W(3, \mathbb{Z}_{3r}))$

Finite ring	regularity $q+1$ first eigenvalue	second eigenvalue	$2\sqrt{q}$	$\|V\|$
$\mathbb{Z}_9$	10	7.3485	6	1640
$\mathbb{Z}_{15}$	16	12.2474	7.746	7232
$\mathbb{Z}_{21}$	22	17.1464	9.1652	19448
$\mathbb{Z}_{27}$	28	22.0454	10.3923	40880

TABLE IX
EXPANDING PROPERTIES OF $G(5, D(4, \mathbb{F}_q))$ AND $G(5, W(4, \mathbb{F}_q))$

Number field	regularity $q+1$ first eigenvalue	second eigenvalue	$2\sqrt{q}$	$\|V\|$
$\mathbb{F}_2$	3	2.7855	2.8284	62
$\mathbb{F}_3$	4	3.4641	3.4641	242
$\mathbb{F}_4$	5	4	4	682
$\mathbb{F}_5$	6	4.4721	4.4721	1562
$\mathbb{F}_7$	8	5.2915	5.2915	5602
$\mathbb{F}_{11}$	12	6.6332	6.6332	32210

Conjecture 1. *The graphs $G(4, D(3, \mathbb{F}_q))$ and $G(4, W(3, \mathbb{F}_q))$ for arbitrary large q are $q+1$-regular Ramanujan graphs and $\lambda_1 \leq 2\sqrt{q}$.*

Conjecture 2. *The graphs $G(5, D(4, \mathbb{F}_q))$ and $G(5, W(4, \mathbb{F}_q))$ for arbitrary large q are $q+1$-regular Ramanujan graphs and $\lambda_1 = 2\sqrt{q}$.*

TABLE X
EXPANDING PROPERTIES OF $G(6, W(5, \mathbb{F}_q))$

Number field	regularity $q+1$ first eigenvalue	second eigenvalue	$2\sqrt{q}$	$\|V\|$
$\mathbb{F}_2$	3	2.8688	2.8284	126
$\mathbb{F}_3$	4	3.8979	3.4641	728
$\mathbb{F}_4$	5	4.4721	4	2730
$\mathbb{F}_5$	6	5.0321	4.4721	7812
$\mathbb{F}_7$	8	5.9541	5.2915	39216

TABLE XI
EXPANDING PROPERTIES OF $G(6, D(5, \mathbb{F}_q))$

Number field	regularity $q+1$ first eigenvalue	second eigenvalue	$2\sqrt{q}$	$\|V\|$
$\mathbb{F}_2$	3	2.9032	2.8284	126
$\mathbb{F}_3$	4	3.3557	3.4641	728
$\mathbb{F}_4$	5	4.8284	4	2730
$\mathbb{F}_5$	6	4.6852	4.4721	7812
$\mathbb{F}_7$	8	5.9228	5.2915	39216

Conjecture 3. *The graphs $G(6, D(5, \mathbb{F}_p))$ and $G(6, W(5, \mathbb{F}_q))$ for arbitrary large q (primr power) and p (prime number) are expanders.*

Conjecture 4. *The graphs $G(4, D(3, \mathbb{Z}_{2r}))$ and $G(4, W(3, \mathbb{Z}_{2r}))$ for arbitrary large r are $2r+1$-regular expander graphs with constant spectral gap $|2r+1-\lambda_1| = 1$.*

The graphs $G(n+1, \Gamma(n, \mathbb{F}_q))$ for arbitrary n, q and any bipartite graph Γ are connected even if Γ is disconnected. What more we have conjecture that the family $G(n+1, \Gamma(n, \mathbb{F}_q))$ is $q+1$-connected, namely highly connected. A graph is said to be k-connected when there does not exist a set of $k-1$ vertices whose removal disconnects the graph.

The connectivity of graphs is important property used in many practical and theoretical aspects.

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