Preprints of the Federated Conference on
Computer Science and Information Systems pp. 41–54

Parthood and Convexity as the Basic Notions of a Theory of Space

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***Abstract*—A deductive system of geometry is presented which is based on atomistic mereology ("mereology with points") and the notion of convexity. The system is formulated in a liberal many-sorted logic which makes use of class-theoretic notions without however adopting any comprehension axioms. The geometry developed within this framework roughly corresponds to the "line spaces" known from the literature; cf. [1, p. 155]. The basic ideas of the system are presented in the article's Introduction within a historical context. After a brief presentation of the logical and mereological framework adopted, a "pregeometry" is described in which only the notion of convexity but no further axiom is added to that background framework. Pregeometry is extended to the full system in three steps. First the notion of a line segment is explained as the convex hull of the mereological sum of two points. In a second step two axioms are added which describe what it means for a thus determined line segment to be "straight". In the final step we deal with the order of points on a line segment and define the notion of a line. The presentation of the geometric system is concluded with a brief consideration of the geometrical principles known by the names of Peano and Pasch. Two additional topics are treated in short sections at the end of the article: (1) the introduction of coordinates and (2) the idea of a "geometrical algebra".**

I. Introduction

Geometry is a very old science and from its very beginnings it was a classical place for discussing the relationship between qualitative and quantitative reasoning. Synthetic geometry as developed in the first four books of Euclid's *Elements* [2] is a paradigm instance of qualitative reasoning. The concept of number is only introduced in Book VII of the work; and the treatment of numbers and the investigation of their properties make use of geometrical representations. As is testified by our parlance about square and cubic numbers, remnants of this procedure are still present in contemporary mathematical terminology. Zeuthen, in his history of ancient mathematics, refers to this procedure as "geometric arithmetic" and "geometric algebra"; cf. [3, pp. 40–53].

The relationship between geometry and algebra was turned around when, in the 17th century, Fermat and Descartes, translating geometric construction tasks into problems concerning the solution of equations, laid the foundations of analytic geometry and thus paved the way for the use of algebra and, later, calculus for the solution of geometric problems. This is rightly considered a major breakthrough in geometric research. However, not so long after Fermat's and Descartes' innovation, already Leibniz argued that the use of (numerical) analysis for achieving geometric results is a detour since analysis is concerned with magnitude and thus only indirectly ("per circuitum") faces such geometric notions as shape ("forma") and similarity ("similitudo"); cf. [4]. According to him, numerical analysis is therefore to be supplemented by a geometric analysis — an *analysis situs* — which deals with such important geometric properties in a direct rather than roundabout way. This geometric analysis is based on a calculus of geometric concepts which makes use of a symbolic language ("characteristica geometrica") resembling that of algebra. Though "analysis situs" became the original, now obsolete name for what is called (general) topology today, there is scarcely a connection between this modern discipline and Leibniz' original ideas.[1] However, Hermann Grassmann, in a treatise submitted as an answer to a prize question asked by a scientific society of Leibniz's hometown Leipzig, re-interpreted Leibniz's ideas in the framework of his "lineale Ausdehnungslehre", which we today consider as a rather abstract and general formulation of vector algebra.[2]

What Leibniz had in mind when he proposed his *analysis situs*, was not a simple return to Euclid's synthetic method and to his deductive procedure but rather an algebraic formulation of geometry in which one could confirm geometric proofs by calculations which directly deal with such geometric entities such as angles, triangles, squares, and circles without first encoding them into numbers, thus translating a geometric problem into one of numerical algebra or analysis. By this he hoped to replace long and intricate arguments to be found in Euclid's *Elements* by simple calculations; cf. the examples given by him at the end of this brief note; [4, pp. 181–183]. Today we are tempted to say that he tried to reduce the

[1] The reference to "position" (Latin *situs*) is not uncommon in geometric research of the 19th century. Thus, for instance, in 1803 the French mathematician L. N. M. Carnot published a book with the title *Géométrie de position*, in which he tried to combine intuitive synthetic geometry with algebraic analytic geometry. Another example is Ch. von Staudt, who in 1847 presented his formulation of projective geometry in a book *Geometrie der Lage*, which is an exact German translation of the title of Carnot's book. Leibniz, when developing his idea of a "characteristica geometrica", might have been acquainted with the geometric ideas of Desargues which led up in the 19th century to the development of projective geometry; cf. [5]. — The 19th century is generally considered "a golden age of geometry"; cf. [6, ch.].

[2] Both works of Grassmann, his *Lineale Ausdehnungslehre* and his *Geometrische Analyse geknüpft an die von Leibniz erfundene geometrische Charakteristik*, have been reprinted in [7].

computational complexity of spatial reasoning. Grassmann, in the treatise mentioned in the previous paragraph, delivers an insightful analysis of Leibniz' first attempts in that direction, cf. [7, pp. 328–334] and puts forwards, in the framework of his own "Ausdehnungslehre", some suggestions for improving Leibniz' work. In order to be able to apply his conceptual framework to geometry, however, he analyses "geometric magnitudes" such as points and line segments as pairs consisting of a geometric entity (a position in the case of a "point magnitude" and a direction in the case of a "line magnitude") and a "metrical value" ("Masswerth"). Hence in his algebraic analysis of Euclid's basic operation of connecting two points by a straight line he re-introduces numbers which Leibniz wished to eliminate; cf. [7, p. 355f].

An analysis of the Euclidean operation of joining two points which is more consonant with Leibniz' original ideas has been given only much later by Walter Prenowitz; cf. [8], [9]. Given (not necessarily different) points p_1, p_2, their join p_1p_2 is the linear segment between them; cf. [9, p. 3].[3] For the join operation, then, algebraic laws such as that of commutativity $p_1p_2 = p_2p_1$, assiociativity $(p_1(p_2p_3) = (p_1p_2)p_3)$, and idempotency $(pp = p)$ are postulated. This looks as if the set of points and the join operation make up an idempotent, Abelian groupoid. However, a closer look upon the first law reveals that this algebraization of a geometric topic rests upon a notational convention. The result p_2p_3 of joining the points p_2 and p_3 is a line segment; but what then is the result of joining this line segment to a point? Prenowitz conceives of line segments as point sets. The range of the binary join operation thus is a set of point sets; furthermore, he declares $(p_1(p_2p_3)$ to be the set of points lying on segments which join the point p_1 with some point of the set p_2p_3. Hence $p_1(p_2p_3)$ is the set of points within the triangle $\triangle p_1p_2p_3$. If we want to conform to absolute exactness, we should either conceive of the join operation as an operation on point sets — and thus formulate associativity by something like $\{p_1\} \cdot p_2p_3 = p_1p_2 \cdot \{p_3\}$ — or we should use a background set theory which identifies urelements with their singletons.[4] Ignoring the distinction between an individual and its singleton set, forces one also to blur the distinction between the relations of membership and inclusion: as an individual the point p_1 is a member of the set p_1p_2, but as its own singleton it is also a subset of that set.

Prenowitz, of course, is completely aware of this; cf. his footnote 5 in [8, p. 3]. Instead of relying on the good instinct of the reader of his writings who restores the set-theoretic distinctions whenever necessary, there would have been an alternative for him, namely to use mereology instead of set theory as a background theory. Given a mereological background, single points of a line segments bear the same relationship to that segment as complete subsegments do: both its points and its subsegments are just parts of the entire segment. The present article follows the strategy just suggested by adopting mereology as a framework for geometry. This issue will be taken up in section II-B below. Adopting mereology "homogenizes" points and segments: both are individuals and the arguments of the join operation are thus on an equal footing. Terms like "$p_1(p_2p_3)$" can be interpreted in a straightforward way which does not require special care of the reader. However, mereology does not resolve our problem completely. It is fine to have both points and line segments as first class citizens of the entire universe of discourse, but these two entities are nevertheless of different kinds. There are things we want to say about points which do not make sense for segments. Since the inception of many-sorted logic systems in the 1930s geometry always has been a prime application area for many-sorted logics; and Arnold Schmidt [11, p. 32], in his classical article on this topic, explicitly refers to Hilbert's axiomatisation of Euclidean geometry [12] in order to motivate the introduction of sortal distinctions. Introducing sortal distinctions between points and segments, however, reintroduces our problem with the interpretation of terms like $p_1(p_2p_3)$ — unless, of course, a more liberal sort system is adopted that allows for the crossing of sort boundaries which is strictly forbidden in such rigid systems as that of Schmidt [11]. Such a liberal system, due to Arnold Oberschelp [13, ch. 3], is adopted in section II below.

The main use Prenowitz makes of the join operation is to define the notion of convexity which is central for his approach to geometry; cf. [9, pp. 25–28]. There is plenty of reason to follow Prenowitz in assigning a central role to the notion of convexity. (1) It plays a central role in various other parts of mathematics as documented in the comprehensive handbook [14]. (2) In quite a few important applications of computational geometry it plays a crucial role; cf. the list given in [15, p. 63]. (3) It seems to be of special importance for the human cognitive systems also in areas beyond geometry; cf. [16, pp. 69–74, 157–174].[5] We shall therefore give convexity a central position in our system of geometry presented below. Its position in that system is even more central than that it occupies in Prenowitz' since we start with the notion of convexity and define that of a linear segment in terms of it whereas definitional dependence in the other way round in Prenowitz' system.[6] However, first our logical

[3]In Prenowitz' 1943 article on this topic, the additive notation "$p_1 + p_2$" is preferred; cf. [8, p. 236]. For Prenowitz, the segment p_1p_2 resulting from joining p_1 and p_2 does not include these two boundary points. Thus, for him, a segment is an "open" set of points. In contrast to this, the join operation which will be defined below in *Def. 14)*-(c) includes the boundary points.

[4]As is done, for instance, in Quine's NF; cf. [10].

[5]Given the importance and usefulness of the notion of convexity, it does not come to a big surprise that it already has made it appearance in formal systems for the representation of spatial knowledge; cf., e. g., [17], [18], and [19]. In [17, sec. 4.3] it is assumed that the convexity function conv which assigns to regions their convex hulls "is only well sorted when defined on one piece regions". No such restriction is assumed here for the hull operator [ ] which will be introduced below in Def 14. The domain of discourse of the interesting theory put forward by [19] is the set of "regular open rational polygons of the real plane" (p. 5). We adopt a much more comprehensive notion of a region (cp. fn. 8) and do not make any decision on the matter of dimensions.

[6]In a strict formal sense, we actually do not define the notion of a segment in terms of convexity. The first notion is present in our system from the start since the many-sorted language used comprises a special sort **s** of segments. However, the axiom *Mer 3* below specifies a sufficient and necessary condition for being a segment.

and mereological background has to be explained in order to prepare the stage for the treatment of these geometric topics.

II. Background: Class Theory and Mereology

The system of geometry proposed here builds upon two more basic formal theories: (1) a certain system of "class logic" and (2) a system of mereology.[7] The version of class logic chosen here is the system LC developed by Arnold Oberschelp [13, chap. 3]. Basically, class logic is "set theory without comprehension axioms". The system LC will be described in more detail in the first part of the present section. Mereology has been used as an ingredient in several axiomatisation of geometric theories; cf., e.g., [23], [24], [25], [26], [27]. The specific system of mereology used here will be introduced in section II-B below.

A. Class Theory

The specific version of LC used here is formulated within a many-sorted language with four sorts denoted by **u**, **c**, **s**, **p**. The universal sort **u** is the sort of all regions. In the present system a region is any mereological sum of points. Thus a region does not need to be connected or three-dimensional, but each region has at least one punctual part; cf. MER 6 below.[8] The remaining three sorts are, respectively, the sort of convex regions (**c**), of (linear) segments (**s**), and of points (**p**). The universal sort contains all other sorts as its subsorts; segments are special convex regions and points, as we shall see below (cf. The 17), special segments. For each sort there are infinitely many variables. We reserve the letter used as the index of a sort for the variables of that sort; thus, e.g., p, p_1, p_2, ... are the variables for points. The letters v, w, v_1, v_2, ... are used as meta-linguistic signs for variables (of any sort); u always refers to a variable of sort **u**. If it is else necessary to indicate a term's membership in a sort s, this will be done by adding "s" as a superscript. Besides the variables there is only one single constant "P" for the part-whole-relation; this constant is an example of a class term and is not assigned to any of the sorts. Semantically LC distinguishes between individuals and objects. Individuals are special objects; they are the possible values of the variables hence the "real" objects. Some objects, however, are not individuals; they are only "virtual", lie without the domain of quantification, and thus do not belong to any sort. LC abstains from any assumptions about the existence of classes and so class terms may denote merely virtual objects. "P" stands for a relation, i.e., a class of pairs of individuals.

There are three groups of logical signs in our version of LC: (1) the connectives $\neg$, $\wedge$, $\vee$ $\rightarrow$, and $\leftrightarrow$; (2) the quantifiers $\exists$ and $\forall$; (3) the relational signs = (identity) and $\in$ (membership); (4) the elementary term constructor $\langle\ ,\ \rangle$ (pair formation); and finally (5) the variable binding term constructors ι (definite description) and $\{\ |\ \}$ (class formation). We use the letters "X" and "Y" as metalinguistic variables for terms, and "φ" and "ψ" for formulas. These two classes of expressions are defined by a simultaneous recursion. (a) Each variable is a term and so is the constant "P". (b) If X and Y are terms, then $\langle X, Y\rangle$ is a term, too. (c) If X and Y are terms, then $X = Y$ and $X \in Y$ are formulas. (d) If φ and ψ are formulas, so are $\neg\varphi$ and $[\varphi \circ \psi]$ where $\circ$ is one of the signs $\wedge$, $\vee$ $\rightarrow$, or $\leftrightarrow$. (e) If φ is a formula and v a variable, then $\exists v.\varphi$ and $\forall v.\varphi$ are formulas and $\iota v.\varphi$ and $\{v \mid \varphi\}$ terms. Terms denote either individuals or classes. In the following we shall use the letters "a" and "b" (possibly with subscripts) for terms of the first kind. For terms denoting classes of individuals we shall use the letters "A" and "B"; finally, the letter "R" is reserved for classes of tuples of individuals.

The logic for the connectives and quantifiers is classical with two exceptions. First, in order to exclude certain trivial cases, LC requires that there are at least two individuals ($\exists u_1 u_1.u_1 \neq u_2$) whereas one postulates in the semantics of standard predicate logic only that the universe of discourse in not empty. The second difference concerns the rule SUB of substitution of free variables by terms. The presence of class terms in LC make it necessary to restrict this rule in order to protect the system against antinomies. In order to formulate the rule, we have first to define the notion of the domain D^s of sort s.

Def 1: (a) $\mathrm{D}^s \underset{\text{def}}{=\!=} \{v^s \mid v^s = v^s\}$

(b) $\mathbb{D} \underset{\text{def}}{=\!=} \mathrm{D}^{\mathbf{u}}$

Let now in the following formulation of the rule SUB $\varphi(v^s)$ be a formula with the free variable v^s and X a term which does not contain any free variable which is bound by a quantifier of $\varphi(x)$ in whose scope v^s occurs as a free variable, then we denote by "$\varphi^{X}_{v^s}$" the result of substituting each free occurrence of v^s in $\varphi(v^s)$ by X. The rule SUB, then, reads as follows.

SUB From $X \in \mathrm{D}^s$ and $\varphi(v^s)$ one may infer $\varphi^{X}_{v^2}$.

The reason for the additional premise becomes obvious as soon as we consider the class theory of LC. It consists of three principles: the principle of extensionality and two abstraction principles.

LC 1: (Ext) $\forall u.[u \in \{v \mid \varphi(v)\} \leftrightarrow u \in \{w \mid \psi(w)\}] \rightarrow \{v \mid \varphi(v)\} = \{w \mid \psi(w)\}$

(Abs$_1$) $v \in \{v \mid \varphi\} \leftrightarrow \varphi$

(Abs$_2$) $X \in \{v^s \mid \varphi\} \rightarrow X \in \mathrm{D}^s$

[7] Mereology is one of the two logical theories which the Polish logician Stanisław Leśniewsik proposed as frameworks for the explication of the traditional notion of a class. Leśniewski discerned two different meanings within that notion, namely that of a "distributive" and that of a "collective" class. Collective classes are treated in mereology, i.e, the theory of the part-of-relationship, whereas distributive classes are the topic of what he called "ontology", the theory of the is-a-relationship. The relationship between common set theory and mereology has been investigated in, e.g., [20] and [21, esp. chs. 5 and 7]. Such a comparison, however, is a delicate issue since Leśniewski based mereology upon his ontology which is a more powerful logic than elementary predicate logic; cf. [22] for a detailed discussion. In the present article, too, (atomistic) mereology is transplanted into a non-Leśniewskian framework.

[8] It should be pointed out here that this is a quite comprehensive (and non-standard) concept of a region. Tarski [23, p. 24] suggests that the "solids" of the geometry envisaged by Leśniewski and the "events" of Whitehead's space-time are "intuitive correlates of open (or closed) regular sets". This (or something like this) seems to be true also for the common systems of mereotopology. The set of points corresponding to a region in the sense explained above in the main text, however, does not need to be regular.

The formation rules allow to build such a term as, e.g., $\{u|u \notin u\}$. It is nevertheless not possible to derive Russell's antinomy by means of (Abs_1) since (Sub) licenses the substitution of $\{u| u \notin u\}$ for the free variable v in $(\mathrm{Abs})_1$ only under the proviso that the class $\{u \mid u \notin u\}$ can be proven to be an individual of sort s. Russell's antinomy shows that this cannot be the case for any s.

The theory of identity contained in LC is quite standard. There are two axioms requiring that identity is reflexive and euclidean. A further axiom finally postulates that identical individuals belong precisely to the same classes.

LC 2: $(\mathrm{Id})_1$ $X = X$
$(\mathrm{Id})_2$ $X = Z \wedge Y = Z \rightarrow X = Y$
$(\mathrm{Id})_3$ $X_1 = X_2 \wedge Y_1 = Y_1 \rightarrow [X_1 \in Y_1 \leftrightarrow X_2 \in Y_2]$

As is common LC construes relations as classes of pairs. There are two axioms for pairs. The first one states the usual criterion of identity for pairs: they are the same iff their components are. The second axiom postulates that pairs of individuals are individuals again.

LC 3: $(\mathrm{Pr})_1$ $\langle X_1, Y_1\rangle = \langle X_2, Y_2\rangle \leftrightarrow [X_1 = X_2 \wedge Y_1 = Y_2]$
$(\mathrm{Pr})_1$ $X, Y \in \mathbb{D} \rightarrow \langle X, Y\rangle \in \mathbb{D}$

The use of the description operator $\imath$ is regulated by three axioms. The first requires that $\imath v.\varphi$ is the individual$v$ if there is exactly one φ and $\varphi(v)$. If there is no unique individual with property φ, then the definite description $\imath v.\varphi(x)$ denotes a special "ersatz" individual $\underline{\perp}$ called "the joker". The joker may be defined by $\underline{\perp} \underset{\mathrm{def}}{=\!=} \imath u_0.u_0 \neq u_0$. It does not belong to the universe of discourse $\mathbb{D}$ but is a virtual object.

LC 4: $(\mathrm{Ds})_1$ $\varphi(v) \wedge \overset{1}{\exists} v.\varphi(x) \rightarrow \imath v.\varphi(v) = v$
$(\mathrm{Ds})_2$ $\neg\overset{1}{\exists} v.\varphi(v) \rightarrow \imath v.\varphi(x) = \underline{\perp}$
$(\mathrm{Ds})_3$ $\underline{\perp} \notin \mathbb{D}$

Though there are no comprehension principles in LC, an elementary theory of classes and relations can be developed which provides most of the means of expressions which became common since the days of Cantor. X is a class if it is identical with its *class part*, i.e., with the class of individuals which are elements of X.

Def 2: $\mathrm{Cls}(X) \underset{\mathrm{def}}{\Longleftrightarrow} X = \{u \mid u \in X\}$

We note that the referents of abstraction terms are always classes; cf. [13, p. 235].

The 1: $\mathrm{Cls}(\{v \mid \varphi\})$

Inclusion relates subclasses to superclasses.

Def 3: $A \subseteq B \underset{\mathrm{def}}{\Longleftrightarrow} \mathrm{Cls}(A) \wedge \mathrm{Cls}(B) \wedge \forall u.[u \in A \rightarrow u \in B]$

The Boolean operations may be defined in the standard way. Relations are classes of n-tuples.

Def 4: $\mathrm{Rel}^n(R) \underset{\mathrm{def}}{\Longleftrightarrow} \mathrm{Cls}(R) \wedge$
$\forall u \in R.\exists u_1 u_2 \ldots u_n.\, u = \langle u_1, u_2, \ldots, u_n\rangle$

As usual, the inverse of a relation results from that relation by inverting the order of its pairs.

Def 5: $R^{-1} \underset{\mathrm{def}}{=\!=} \{\langle u_1, u_2\rangle \mid \langle u_2, u_1\rangle \in R\}$

The two definitions below introduce abbreviations used in the following.[9]

Def 6: (a) $R^{>}a \underset{\mathrm{def}}{=\!=} \{u \mid \langle u, a\rangle \in R\}$
(b) $R^{<}a \underset{\mathrm{def}}{=\!=} \{u \mid \langle a, u\rangle \in R\}$

Functions are defined as special relations fulfilling a uniqueness condition:

Def 7: $\mathrm{Fct}^n(R) \underset{\mathrm{def}}{\Longleftrightarrow} \mathrm{Rel}^{n+1}(R) \wedge$
$\forall u_1 \ldots u_{n+2}.[\langle u_1, \ldots u_n, u_{n+1}\rangle,$
$\langle u_1, \ldots, u_n, u_{n+2}\rangle \in R \rightarrow u_{n+1} = u_{n+2}]$

Functions will often be defined by specifying how to determine the value a for given arguments $u_1,\ u_2,\ \ldots, u_n$. Given a certain (n-place) function f, the term "$f(a_1, a_2, \ldots, a_n)$" will denote the value of f for the arguments $a_1,\ a_2,\ \ldots, a_n$ (if it exists).

Def 8: (a) $\lambda u_1 u_2 \ldots u_n.a \underset{\mathrm{def}}{=\!=} \{\langle u_1, u_2, \ldots, u_n, u\rangle \mid u = a\}$
(b) $f(a_1, a_2, \ldots, a_n) \underset{\mathrm{def}}{=\!=} \imath u.\langle a_1, a_2, \ldots, a_n, u\rangle \in f$

Functions may be partial. In mereology, for instance, the *product* $u_1 \cdot u_2$ of two individuals is the largest individual (*modulo* the part-of-relation) which is a common part of both u_1 and u_2. If in a formal system of mereology the product operation $\cdot$ is not taken as primitive, it will be defined by some function involving a definite description; cf., e.g., [28, p. 43]. That description term will be improper if the items denoted by the "factor" terms do not overlap. Since no product exists in this case, the product operation is partial. In LC the product of non-overlapping regions u_1 and u_2 equals the joker: $u_1 \cdot u_2 = \underline{\perp}$.

B. Atomistic Mereology

Whitehead [29]–[31] motivates his use of mereological concepts as a foundation for his space-time-geometry by the desire for a conceptual framework which directly relates this science to spatial reality rather than starting from abstractions such as, e.g., extensionless points and breadthless lines. Using his "method of extensive abstraction", he constructed such entities from extended regions. The critique of such notions as that of a point and that of a line, however, is much older and in fact nearly as a old as the science of geometry itself.[10] In the 19th century, Lobacevski and Bolyai did not only replace Euclid's Fifth Axiom (on the unique existence of parallels) by other assertions but also suggested to take the notion of a

[9]The notation introduced in *Def 6* is often used to render formulas more easily readable. E.g., "$u_1 \in \mathrm{P}^{>}u_2$" can be read from left to right as "u_1 is ($\in$) a part of ($\mathrm{P}^{>}$) u_2". The same is said by "$\langle u_1, u_2\rangle \in \mathrm{P}$", which however requires the reader to apply a "forth and back" procedure when decoding the formula. — "$u_1 \in \mathrm{P}^{<}u_2$" may be read as "$u_1$ extends / is an extender of u_2".

[10]Cf., for instance, Aristotle's remark in his *Metaphysics*, [32, p. 36, 992[a] 20] that Plato "fought against [the kind of points] as being a geometric dogma" and Proclus Lycaeus warning — in his commentary on the first book of Euclid's *Element* — not to follow the Stoics who suppose that such limiting elements like points "exist merely as the product of reflection"; [33, p. 71].

rigid, three-dimensional body as the conceptual starting point for geometry.[11] In a similar vein, Whitehead [29]–[31], relying on a certain analysis of the role of abstractions in science, developed a theory of events which bears some similarities to Leśniewski's mereology, which Tarski [23] combined with ideas of the Italian mathematician Mario Pieri in order to formulate his geometry of solid bodies.[12] This line of research has been continued by the work of Gruszczyński and Pietrusczcak [26]. Whitehead's approach, especially as modified in his book from 1929, has been continued in "mereotopology"—cf., e.g., [28] and [27]—and in work on the Region-Connection-Calculus (RCC); cf. [38] and the literature cited there.

Hahmann *et al.* [27, p. 1424] formulate the objections against points in a concise way: "Points are somewhat tricky to define and are far from intuitive in real-world applications." It is certainly true that the definition of points as equivalence classes of converging sequences of regions (as suggested by Whitehead and others) is "tricky".[13] However, if points are admitted from the outset as special ("extensionless") regions, it is rather easy to single them out by a definition. Actually, we find an adequate explanation already as the very first definition in the first book of Euclid's *Elements*: "A point is that which has no part"; [2, Book I, p. 153]. As is evident from this definition, Euclid obviously is thinking of the proper (irreflexive) part-of-relationship when he is talking about parts. Allowing, as is usual in mereology, also for improper parts, we can reformulate Euclid's definition as follows: "A point is a minimum of the part-of-relation."

Euclid's definition testifies that the notion of a point nicely fits into a mereological framework. Aside from this formal issue, perceptual psychology does not seem to support the sceptical attitude towards points held up by many supporters of "common sense geometry". Experimental studies of visuals space simply accept the existence of points when they approximate these geometric items by "small point sources of light of low illumination intensity, displayed in darkened room;" [46, p. 238]. Points seem also to be accepted in phenomenological and gestalt-theoretic approaches to psychology. In a series of classical experiments Edgar Rubin [47, §§14–16] showed that points (as well as other regions lacking extension in one or more dimension) are really perceived: "As there are breadthless lines, there are extensionless points". Furthermore, Otto Selz [48, p. 40] argued that points essentially belong to our conceptual frame used in the apprehension of space: "the pure location in space is postulated by structural laws in the same way as the infinity of the straight line and [...] it is of relatively minor importance whether the empirical *Minimum Visibile*, i.e., the point gestalt, is to be regarded as a pure locational phenomenon or rather as a tiny round area like object". We hence conclude that points, though they are perhaps no "real constituents" of physical space, do have perceptual reality and exist in conceptualized space. This is all which is of importance in the present context.

As the mereological foundation of our system of geometry we adopt the system of atomistic mereology developed by Tarski; cf. [49]. Tarski formulated his system within the simple theory of types. Instead we use the class logic LC sketched in the previous subsection. The only undefined notion in Tarski's system is the relation P of parthood[14] of which it is postulated that it is transitive. In LC this correspond to the following two axioms.

MER 1: $\mathrm{Rel}^2(\mathrm{P})$
MER 2: $\langle u_1, u_2\rangle, \langle u_2, u_3\rangle \in \mathrm{P} \rightarrow \langle u_1, u_3\rangle \in \mathrm{P}$

We say that two individuals (regions) overlap if they share a common part.[15]

Def 9: $\mathrm{O} \underset{\text{def}}{=\!=} \{\langle u_1, u_2\rangle \mid \mathrm{P}^{>}u_1 \cap \mathrm{P}^{>}u_2 \neq \emptyset\}$

The formulation of the next axioms requires the following definition.

Def 10: $\Sigma(a, A) \underset{\text{def}}{\Longleftrightarrow} A \subseteq \mathrm{P}^{>}a \wedge \forall u_1 \in \mathrm{P}^{>}a.\exists u_2 \in A.\langle u_1, u_2\rangle \in \mathrm{O}$

The formula "$\Sigma(a, A)$" says that a is the mereological sum of the individuals in A. This means that every element of A is a part of a and that conversely every part of a overlaps some element of A. The mereological sum of a singleton class is the unique member of that class; and non-empty classes always have a sum.

MER 3: $\Sigma(u_1, \{u_2\}) \rightarrow u_1 = u_2$
MER 4: $A \neq \emptyset \rightarrow \exists u.\Sigma(u, A)$

From the axioms stated until now it can be proven[16] that P is a partial order of the elements of $\mathbb{D}$, i.e., that the part relation, besides being transitive, is reflexive and antisymmetric. Furthermore, *MER 4* may be strengthened by asserting the uniqueness of the mereological sum.

[11]Their endeavors are described in Richard Strohal's investigations of the relationships between "pure geometry" and intuition; cf. [34, pp. 20–33]. It deserves to be mentioned that Lobacevski considered the relationship of connection (between "solids") as the most basic concept of geometry thus anticipating the modern line of research which starts with the work of de Laguna [35] and leads up via Whitehead's reformulation of his earlier work in [31] to the Region Connection Calculus of Randell, Cui, and Cohn [17].

[12]In his lecture notes [36], Leśniewski compares his mereology with Whitehead's theory of events. In those notes, Leśniewski mentions that it was Tarski in 1926 who made him aware of Whitehead's work; cf. [36, p. 171]. — Pieri's idea employed by Tarski [23] in his mereological system of geometry is that this discipline can be developed starting from the notions of point and sphere as the only undefined concepts. Pieri's memoir presenting this idea has been re-published in an English translation by Marchisotto and Smith [37, pp. 157–288].This book contains also a chapter on Pieri's impact on Tarski's geometric work; cf. [37, ch. 6]. That the notion of sphere is sufficient as a basic concept of geometry has been noted already by Grassmann in his 1847 memoir on Leibniz' geometric analysis; cf. [7, p. 328].

[13]The issue of "region-based" vs. "point-based" geometry is treated in quite a few articles on mereotopology; cf. [39] and [40], who both provide surveys of the classical approaches to this topic by Whitehead [29]–[31], De Laguna [35], Menger [41], Grzegorczyk [24], and Clarke [25]. More recent contributions include [42], [43], [27], [44], and [45].

[14]In [49] Tarski augments his system of pure mereology by other non-mereological systems in order to make it suitable as a basis for axiomatic biology.

[15]The notion of overlap is not used by Tarski. We introduce it here in order to make our presentation more similar to standard expositions of mereology; cf., e.g., [28].

[16]For the proofs of the mereological theorems the reader is referred to Tarski's article [49].

The 2: (a) $\langle u, u\rangle \in \mathrm{P}$
(b) $\langle u_1, u_2\rangle, \langle u_2, u_1\rangle \in \mathrm{P} \rightarrow u_1 = u_2$
(c) $A \neq \emptyset \rightarrow \overset{1}{\exists} u.\Sigma(u, A)$

The 2-(c) justifies the following definitions introducing the notion of a supremum or sum of a class of individuals.

Def 11: (a) $\sup(A) \underset{\text{def}}{=\!=} \iota u.\Sigma(u, A)$
(b) $\sup(v^s \mid \varphi) \underset{\text{def}}{=\!=} \sup(\{v^s \mid \varphi\})$
(c) $\sup(a_1, a_2, \ldots, a_m) \underset{\text{def}}{=\!=} \sup(\{a_1, \ldots, a_m\})$
(d) $+ \underset{\text{def}}{=\!=} \lambda u_1 u_2. \sup(u_1, u_2)$

Of course, we shall always write "$a + b$" instead of "$+(a, b)$". It can be proven that the thus defined notion of a supremum of A has indeed the properties normally required: namely, that it is the "smallest" individual "bigger" than all the elements of A, cf. *The 3*.

The 3: $u = \sup(A) \rightarrow A \subseteq \mathrm{P}^{>}u \wedge \forall u_1.[A \subseteq \mathrm{P}^{>}u_1 \rightarrow u \in \mathrm{P}^{>}u_1]$

It is provable in LC that $\mathbb{D}$ is non-empty; hence by *The 2*-(c) there exists the sum of all individuals. Following [49, p. 162], we shall call it $\mathbf{w}$ (which Tarski transliterates as "world"). It is the entire space.

Def 12: $\mathbf{w} \underset{\text{def}}{=\!=} \sup(\mathbb{D})$

The space is an individual, hence its exists, and everything, i.e., every region, is a part of it; cf. *The 4*.[17]

The 4: (a) $\mathbf{w} \in \mathbb{D}$
(b) $u \in \mathrm{P}^{>}\mathbf{w}$

Corresponding to the notion of the mereological sum of a class of individuals there is the notion of a product. This is not defined by Tarski; but *Def13* suggests itself by its analogy to the case of the sum.

Def 13: (a) $\Pi(a, A) \underset{\text{def}}{\Longleftrightarrow} \Sigma(a, \{u \mid \forall u_1 \in A.\langle u, u_1\rangle \in \mathrm{P}\})$
(b) $\inf(A) \underset{\text{def}}{=\!=} \iota u.\Pi(u, A)$
(c) $\inf(v^s \mid \varphi) \underset{\text{def}}{=\!=} \inf(\{v^s \mid \varphi\})$
(d) $\inf(a_1, a_2, \ldots, a_m) \underset{\text{def}}{=\!=} \inf(\{a_1, a_2, \ldots, a_m\})$
(e) $\cdot \underset{\text{def}}{=\!=} \lambda u_1 u_2. \inf(u_1, u_2)$

Again we use infix notation "$a_1 \cdot a_2$" instead of "$\cdot(a_1, a_2)$". There is an important difference between the notion of a sum and that of a product: whereas only non-empty classes have a sum, also the empty class has a product. If $A = \emptyset$, then $\{u \mid \forall u_1 \in A.\langle u, u_1\rangle \in \mathrm{P}\} = \mathbb{D}$ and hence $\Pi(A, \mathbf{w})$ according to *Def 12*. This, however, does not mean that the infimum always exists. If A is a class of non-overlapping individuals, i.e., of individuals which have no common parts, then the class $\{u|\forall u_1 \in A.\langle u, u_1\rangle \in \mathrm{P}\}$ will be empty and will hence not have a supremum. In this case, therefore, A will not have an infimum.

[17]Theorems like *The 4*-(a) are of special importance for our formal framework since the rule of substitution of LC is, as has been explained in section II-A above, restricted in such a way that the substitution of a term X for a variable of sort s requires a proof of $X \in \mathrm{D}^s$.

Corresponding to *The 3* we have the following theorem for the infimum.

The 5: $u = \inf(A) \rightarrow A \subseteq \mathrm{P}^{<}u \wedge \forall u_1.[A \subseteq \mathrm{P}^{<}u_1 \rightarrow u_1 \in \mathrm{P}^{<}u]$

According to *The 4*, $\mathbf{w}$ is the unique region which is maximal with respect to the P-relation. Now after we have decided to adopt points as the minima of that relation, it is useful also to adopt a special sort $\mathbf{p}$ for points. Hence $\mathrm{D}^{\mathbf{p}}$ (cf. *Def 1*-(a)) is the class of all points which thus does not need a special definition. However, in order to catch the identification of points with P-minima, we have to accept a special axiom which corresponds to Tarski's Definition of points; cf. [49, p. 163].

MER 5: $\mathrm{D}^{\mathbf{p}} = \{u \mid \mathrm{P}^{>}u \subseteq \{u\}\}$

MER 5 has still to be supplemented by Tarski's postulate that each individual has at least one punctual part.

MER 6: $\mathrm{P}^{>}u \cap \mathrm{D}^{\mathbf{p}} \neq \emptyset$

MER 6 is sufficient to show that each individual is the sum of its points; Tarski's proof for this can be transferred to the present system.

The 6: $u = \sup(p \mid p \in \mathrm{P}^{>}u)$

However, within the framework of LC this does not mean that talk about regions can be dismissed in favour of talk about point classes since within LC (unlike as in Tarski's type-theoretic framework) we cannot quantify over point classes though quantification over regions is possible.

III. Interval Spaces and Convexities

In the previous sections we have laid the logical and mereological foundations for the system of geometry which will be presented in a stepwise manner in this and the following two sections. In the first part of the present section, we do not extend the foundational framework provided by any further axioms but define some concepts of central importance for our system of geometry. Then we point out some simple consequences which can be derived from the definitions given only by means of logic and mereology. In the second part of the present section we then state the first axioms of a geometric character.

A. Pregeometry

By a a convex region we understand a region in which every pair of points is connected by a linear segment completely belonging to that region. A triangle and a circle are examples of convex regions whereas the bean shaped region of Fig. 1 is not. The variables of sort $\mathbf{c}$ vary over the elements of the domain $\mathrm{D}^{\mathbf{c}}$ (cf. Def. 1) which is the class of all convex regions. *Def 14* introduces the central notion of the *convex hull* of a region: the function [ ] assigns to each region u its convex hull $[\,](u)$. We write "$[u]$" instead of "$[\,](u)$" in order to comply with ordinary notation. The convex hull $[u]$ is the infimum of all convex regions containing u as a part.

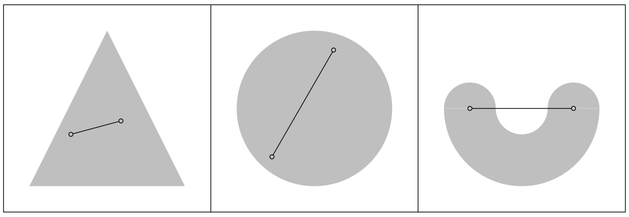

Fig. 1. Two convex regions and a non-convex one

Def 14: (a) $[\,] \underset{\text{def}}{=\!=} \lambda u.\inf(c \mid u \in \mathrm{P}^{>}c)$

(b) $[p_1, p_2, \ldots, p_n] \underset{\text{def}}{=\!=} [\sup(p_1, p_2, \ldots, p_n)]$

(c) $p_1 p_2 \underset{\text{def}}{=\!=} [p_1, p_2]$

Def 14-(b) defines the *polytope* — or, more precisely, the *n*-tope — spanned by p_1, p_2, …, p_n as the convex hull of the sum of those points. The *segment* p_1p_2, then, between p_1 and p_2 is just the 2-tope spanned by these two points, cf. *Def 14*-(c). These definitions correspond to those given in point set based convex geometry; cf., e .g., [1, pp. 3, 5]. The convex hull $[u]$ of a region u always exists (is an individual) and contains u as a part.

The 7: (a) $[u] \in \mathbb{D}$

(b) $u \in \mathrm{P}^{>}[u]$

Proof: Let $A := \{u_1 \mid \forall c.[\langle u, c\rangle \in \mathrm{P} \rightarrow \langle u_1, c\rangle \in \mathrm{P}\}]$. If there are no convex regions containing u, then $A = \mathbb{D}$ and $[u] = \mathbf{w} \in \mathbb{D}$; and (b) holds by *The 4*. If, however, there are convex regions containing u, then $u \in A \neq \emptyset$ and hence $[u] \in \mathbb{D}$ by *The 2*-(c). The assertion (b), then, follows by *The 5*. ∎

From *The 7* we have immediately *The 8*-(a); the second claim of that theorem is a direct consequence of the definition of the product and the hull operation.

The 8: (a) $p_1, p_2 \in \mathrm{P}^{>}p_1p_2$

(b) $p_1p_2 = p_2p_1$

The 8 states (*modulo* the replacement of set theoretical notions by mereological ones) that the class of points and that of segments together with the segment operation constitutes an *interval space*; cf. [1, chap. 1, Sec. 4].[18] *The 8*-(a) corresponds to the so-called *extensive law*, *Theorem 8*-(b) to the *symmetry law* of interval spaces. Interval spaces in turn give rise to *convex structure* (also briefly called *convexities*). These are the structures exhibiting the basic facts about convex sets; cf. [1, p. 3]. A convexity is a family C of subsets of some point set X which fulfills the following three closure conditions:

(C-1) $\emptyset, X \in C$;

(C-2) for $\mathcal{D} \subseteq C$ is $\bigcap \mathcal{D} \in C$;

(C-3) if for $A, B \in \mathcal{D} \subseteq C$ it always holds true that $A \subseteq B$ or $B \subseteq A$, then $\bigcup \mathcal{D} \in C$.

[18]On the background of set theory, an interval space $I = \langle X, I\rangle$ is defined to be a pair consisting of a set X of points and an operation $I : X \times X \rightarrow 2^X$ such that for $p, q \in X$ it holds true that $p, q \in I(p, q)$ and $I(p, q) = I(q, p)$; cf. [1, p. 71].

Using the segment operation, we define in our mereological context the special class Cv of regions in the following way.

$$\mathrm{Cv} \underset{\text{def}}{=\!=} \{u \mid \forall p_1, p_2 \in \mathrm{P}^{>}u.p_1p_2 \in \mathrm{P}^{>}u\}$$

It is not too difficult to show that Cv fulfills mereological analogues to (C-1), (C-2), and (C-3).

The 9: (C-1)' $\mathbf{w} \in \mathrm{Cv}$

(C-2)' $\exists u.A \subseteq \mathrm{P}^{<}u \wedge A \subseteq \mathrm{Cv} \rightarrow \inf(A) \in \mathrm{Cv}$

(C-3)' $\emptyset \neq A \subseteq \mathrm{Cv} \wedge$
$\forall u_1, u_2 \in A.\langle u_1, u_2\rangle \in \mathrm{P} \cup \mathrm{P}^{-1} \rightarrow$
$\sup(A) \in \mathrm{Cv}$

Proof: (C-1)' is immediate from *The 4*-(b). — (C-2)'. The first conjunct of the hypothesis ensures that $u_1 := \inf(A) \in \mathbb{D}$. It remains to be shown that each segment p_1p_2 where $p_1, p_2 \in \mathrm{P}^{>}u_1$ is itself a part of u_1. From $p_1, p_2 \in \mathrm{P}^{>}u_1$ it follows by *The 5* that for each $u_2 \in A$ $p_1, p_2 \in \mathrm{P}^{>}u_2$ and hence $p_1p_2 \in \mathrm{P}^{>}u_2$ since $A \subseteq \mathrm{Cv}$. Thus $p_1p_2 \in u_2$ for each $u_2 \in A$, hence $u_1 \in \mathrm{Cv}$. — (C-3)'. Since $A \neq \emptyset$, again $u_1 := \sup(A) \in \mathbb{D}$. Suppose $p_1, p_2 \in \mathrm{P}^{>}u_1$. According to *Def 10*, the two points share, respectively, a part with two individuals $u_2, u_3 \in A$. According to the second conjunct of the assumption $\langle u_2, u_3\rangle \in \mathrm{P}$ or, conversely, $\langle u_3, u_2\rangle \in \mathrm{P}$. Assume the first (the argument for the second is completely parallel). Then $p_1, p_2 \in \mathrm{P}^{>}u_2$ and, since $u_2 \in A \subseteq \mathrm{Cv}$, $p_1p_2 \in \mathrm{P}^{>}u_2$. But then $p_1p_2 \in \mathrm{P}^{>}u_1$, too. Hence $u_1 = \sup(A) \in \mathrm{Cv}$. ∎

A region u belongs to Cv if all the "2-topes", i.e., segments, whose boundary points are from u lie within that very region. Of course, the definition of the class Cv is an exact formal counterpart of the intuitive explanation of the notion of a convex region provided at the beginning of this subsection. Therefore it cannot be included as a formal definition within our system since this would involve a circularity: segments are defined in terms of convex regions (by using variables of sort **c**), hence one cannot use segments in order to define convex regions. However, the class Cv should turn out to be identical with the domain $\mathrm{D}^{\mathbf{c}}$. Within pregeometry we can prove at least the inclusion of that domain in Cv; cf. *The 10*. The converse inclusion will be postulated as an axiom in the next subsection.

The 10: $\mathrm{D}^{\mathbf{c}} \subseteq \mathrm{Cv}$

Proof: This follows readily from *The 5* and *Def 14*. ∎

B. Convex Structure

Spelled out, the converse of *The 10* amounts to the following principle.

GEO 1: $\forall p_1, p_2 \in \mathrm{P}^{>}u.p_1p_2 \in \mathrm{P}^{>}u \rightarrow u \in \mathrm{D}^{\mathbf{c}}$

By *GEO 1* we leave mereology and pregeometry and enter the realm of geometry proper. Therefore the label "*GEO*" is given to the new axiom rather than continuing using "*MER*" in order to mark principles. By *The 10* we may strengthen *GEO 1* to a biconditional.

The 11: $u \in \mathrm{D}^{\mathbf{c}} \leftrightarrow \forall p_1, p_2 \in \mathrm{P}^{>}u.p_1p_2 \in \mathrm{P}^{>}u$

Furthermore, we may now replace "Cv" in *The 10* by "$\mathrm{D}^{\mathbf{c}}$".

The 12: (a) $\mathbf{w} \in \mathrm{D}^{\mathbf{c}}$
(b) $\exists u.A \subseteq \mathrm{P}^{<}u \wedge A \subseteq \mathrm{D}^{\mathbf{c}} \rightarrow \inf(A) \in \mathrm{D}^{\mathbf{c}}$
(c) $\emptyset \neq A \subseteq \mathrm{D}^{\mathbf{c}} \wedge \forall u_1, u_2 \in A.\langle u_1, u_2 \rangle \in \mathrm{P} \cup \mathrm{P}^{-1} \rightarrow \sup(A) \in \mathrm{D}^{\mathbf{c}}$

The 11 and *The 12* state that $\mathrm{D}^{\mathbf{c}}$ is a (mereological) convex structure. *The 12*-(b) immediately implies that convex hulls are, as their name suggests, convex. A corollary of this is that segments, which are special convex hulls — namely convex hulls of regions consisting of at most two points — are convex.

The 13: (a) $[u] \in \mathrm{D}^{\mathbf{c}}$
(b) $\mathrm{D}^{\mathbf{s}} \subseteq \mathrm{D}^{\mathbf{c}}$

The 13-(a) implies that the hull of a region's hull equals that hull and that the hull operation is monotonic.[19]

The 14: (a) $[[u]] = [u]$
(b) $u_1 \in \mathrm{P}^{>}u_2 \rightarrow [u_1] \in \mathrm{P}^{>}[u_2]$
(c) $u \in \mathrm{P}^{>}c \rightarrow [u] \in \mathrm{P}^{>}c$

Proof: By *The 7*-(b), *The 5*, and *The 13*. ■

The 7-(b) and *The 14*-(a), (b) state that the [ ]-operator is a hull-operator in the algebraic sense. As a special case of *The 14*-(b) we have that segments spanned by the points of some given segment are subsegments of that segment and that hence a point of a given segment dissects a subsegment of that segment.

The 15: (a) $p_1, p_2 \in \mathrm{P}^{>}s \rightarrow p_1p_2 \in \mathrm{P}^{>}s$
(b) $p_3 \in \mathrm{P}^{>}p_1p_2 \rightarrow p_1p_3 \in \mathrm{P}^{>}p_1p_2$

The 15-(b) is called the *monotone law* in [1, p. 74]. — We did not require that the two arguments of the segment operator are distinct. If we ask for the segment $[pp]$ joining the point p to itself, a natural answer would be that in this case the segments shrinks down to the point p. Hence points are just segments without any extension. Points are also special regions: they are the minimal regions. But then, we have to admit that the only segment which is a part of a minimal region p is p itself and that therefore p is convex according to our intuitive explanation of convexity. That this is actually the case is postulated by a new axiom which states that the points are a *subsort* of the convex regions.

GEO 2: $\mathrm{D}^{\mathbf{p}} \subseteq \mathrm{D}^{\mathbf{c}}$

From *GEO 2* it is immediate that points are minimal segments. This is, for obvious reasons, called the *idempotent law* in the theory of interval spaces; cf. [1, p. 74].

The 16: $pp = p$

Proof: From *GEO 2* together with *The 5* and *The 7*. ■

The domain $\mathrm{D}^{\mathbf{s}}$ is characterized by the following axiom.

[19]Of the five axioms stated by Randell et. al. [17, p. 5] for their operator conv, the first one corresponds to The 7-(b) and the second to The 14-(a). The third axiom follows easily from The 14-(b) (in combination with the (a)-clause of that theorem). The two remaining axioms which relate the concept of a convex hull to the relation O (of overlap) and its complement, are true in the "intended model" of the present theory, too. A proof of them, however, is by no means obvious.

GEO 3: $\mathrm{D}^{\mathbf{s}} = \{u \mid \exists p_1, p_2.u = p_1p_2\}$

From *The 16* and the new axiom *GEO 3* it follows that points are special segments, namely one-point-only segments.[20]

The 17: $\mathrm{D}^{\mathbf{p}} \subseteq \mathrm{D}^{\mathbf{s}}$

Another consequence of *The 16* is that each region is the sum of its segmental parts.

The 18: $u = \sup(s \mid s \in \mathrm{P}^{>}u)$

Proof: By *The 6* and *The 16* a region is already the sum of its punctual segments. The non-punctual elements of the class $\{s \mid s \in \mathrm{P}^{>}u\}$ do not add anything more to the mereological sum of this class. ■

In the case of convex regions *The 18* can be given the following strengthened form.

The 19: $p_1 \in \mathrm{P}^{>}c \rightarrow c = \sup(s \mid \exists p_2 \in \mathrm{P}^{>}c.s = p_1p_2)$

Proof: From $p_1 \in \mathrm{P}^{>}c$, it follows by *The 11*, that $p_1p_2 \in \mathrm{P}^{>}c$ for each $p_2 \in \mathrm{P}^{>}c$. Hence $\sup(s|\exists p_2 \in \mathrm{P}^{>}c.s = p_1p_2) \in \mathrm{P}^{>}c$. — It remains to be shown that also conversely $c \in \mathrm{P}^{>} \sup(s|\exists p_2 \in \mathrm{P}^{>}c.s = p_1p_2)$. Assume so that $p_3 \in \mathrm{P}^{>}c$. It suffices to show that $p_3 \in \mathrm{P}^{>} \sup(s \mid \exists p_2 \in \mathrm{P}^{>}c.s = p_1p_2)$. But this follows readily from *The 16*. ■

To conclude the present subsection, we state a further axiom which strengthens the theorem just proven for a special kind of convex regions. Consider some point p and a convex region c. The region $[p + c]$ may be called the *cone* with *apex p* and *base c*; cf. Fig. 2. Since the cone has been constructed as a hull, it is convex. By *The 19*, then, it equals the sum of all the segments starting from the apex and ending at some other point of the cone. The next axiom states that we do not really need to consider all segments of the kind described but rather can restrict ourselves to segments from the apex to the points of the base (as the points p_1, p_2 and p_3 in Fig. 2).

GEO 4: $[p_1 + c_1] = \sup(s_1 \mid \exists p_2 \in \mathrm{P}^{>}c_1.s_1 = p_1p_2)$

This axiom is called *join-hull commutativity* since it postulates that the hull operation and the sum ("join") operation may be interchanged; cf. [1, p. 39]. The reader should remember here that pp_1 actually is the convex hull $[p_1, p_2]$. To make thus the name of the principle more transparent, we could render it as $[\sup(p_1, c_1)] = \sup([p_1p_2] \mid p_2 \in \mathrm{P}^{>}c_1)$.[21]

IV. Straightness and Order

In the previous section we dealt with the relationship between segments and convex regions. Segments connect the points of a convex region without leaving that region. In the present section we shall consider two further important properties of segments. In the first subsection we shall set up two axioms which make explicit what it means for a segment to be "straight" rather than "bent". Then we shall study the order of points in a segment.

[20]Though *The 17* seems to be quite trivial, its formal proof requires some care as regards the handling of the sorts.

[21]Where "$\sup([p_1p_2] \mid \varphi)$" abbreviates "$\sup(s_1 \mid s_1 = p_1p_2 \wedge \varphi\}$".

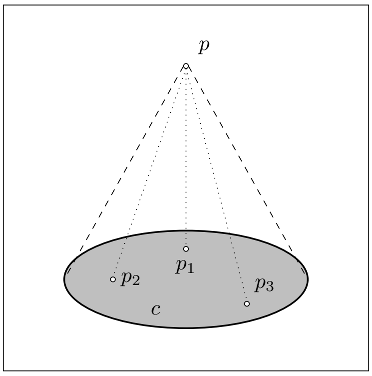

Fig. 2. Join-hull commutativity.

A. Straightness

Segments are convex, as we have seen, and so are the two first sample regions (the triangle and the circle) displayed in Fig. 1. Segments are one-dimensional and thus differ from triangles and circles which are two-dimensional. There is yet another property which sets segments apart from circles. A circles (by its circumference) involves curvature whereas a segment is *straight*. In the present subsection we set up two axiomatic principle which make explicit what it means to be straight. The first of these two principles is known as *decomposability*; cf. [1, p. 143]. In our framework it may be rendered as follows.

GEO 5: $p_2 \in \mathrm{P}^{>} p_1 p_3 \quad \rightarrow \quad p_1 p_3 = p_1 p_2 + p_2 p_3 \wedge$
$p_2 = p_1 p_2 \cdot p_2 p_3$

A point of a segment dissects the whole segment into two component segments which overlap precisely in the dissecting point; cf. the left diagram of Fig. 3. Hence a curved line with a loop such as that displayed by the right diagram of Fig. 3 cannot be a segment since there is a point on that line which dissects it into three parts.

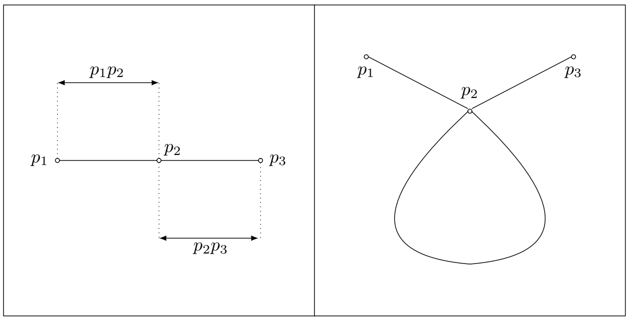

Fig. 3. Decomposition of a segment into two segmental components

As an immediate consequence of decomposability we have:

The 20: $p_2 \in \mathrm{P}^{>} p_1 p_2 \wedge p_3 \in \mathrm{P}^{>} p_1 p_2 \rightarrow p_2 = p_3$

Proof: By decomposability $p_1 p_3 = p_1 p_2 + p_2 p_3$ with $p_2 = \inf(p_1 p_2, p_2 p_3)$. But since $p_3 \in \mathrm{P}^{>} p_1 p_2 \cap \mathrm{P}^{>} p_2 p_3$, $p_2 = p_3$. ■

The second postulate which explains what it means for a line to be straight is known by the name of this property, i.e., *straightness*; [1, p. 143].

GEO 6: $\exists p_1 p_2 . [p_1 \neq p_2 \wedge p_1, p_2 \in \mathrm{P}^{>} s_1 \cap \mathrm{P}^{>} s_2] \rightarrow$
$s_1 + s_2 \in \mathrm{D}^{\mathbf{s}}$

The sum of two segments sharing two points cannot result in a curved line because in that case at least one of the items combined would have already been bent; cf. Fig. 4.

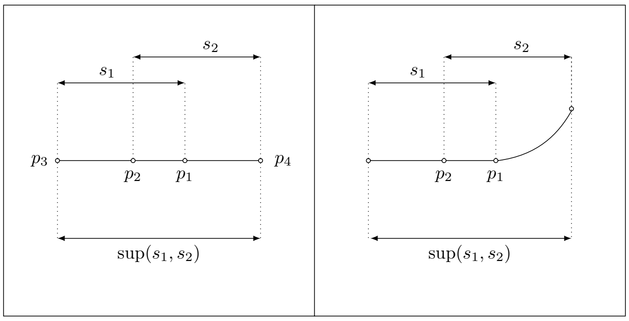

Fig. 4. The combination of two segments results in a straight segment again

The 5 and *The 6* imply the *ramification* principle of *The 21*; cf. [1, p. 143] which says that two segments which have one boundary point in common but differ with respect to their second boundary will branch away from each other at the common point; cf. the left hand side of Fig. 5. The indirect proof of the ramification principle provided by [1, p. 144] within a set-theoretic framework can be directly transferred to our mereological system.

The 21: $p_3 \notin \mathrm{P}^{>} p_1 p_2 \wedge p_2 \notin \mathrm{P}^{>} p_1 p_3 \rightarrow p_1 = p_1 p_2 \cdot p_1 p_3$

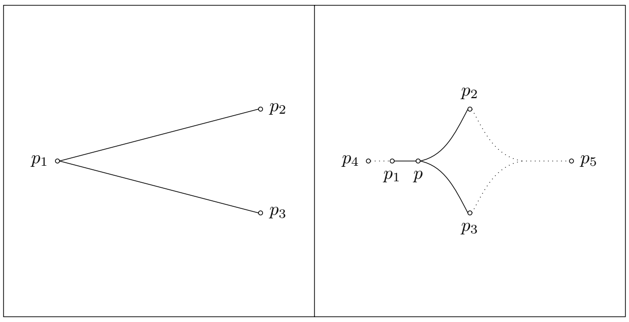

Fig. 5. The Ramification Property.

B. Order and Lines

Normally an order relation between points belongs to the undefined concepts of standard axiomatic systems of geometry; cf., e.g., [12, §3], [50, pp. 11–13]. In our framework such a relation may be defined.

Def 15: (a) $\mathrm{B} \underset{\text{def}}{=\!=} \{\langle p_1, p_2, p_3 \rangle \mid p_2 \in \mathrm{P}^{>} p_1 p_3\}$
(b) $p_2 \mathrm{B} p_1 p_3 \underset{\text{def}}{\iff} \langle p_1, p_2, p_3 \rangle \in \mathrm{B}$

Decomposability implies that of three points at least one lies between the two others.

The 22: $p_1, p_2, p_3 \in \mathrm{P}^{>}s \rightarrow$
$p_1\mathrm{B}p_2p_3 \vee p_2\mathrm{B}p_1p_3 \vee p_3\mathrm{B}p_1p_2$

Proof: If two of the three points mentioned in the assumption of the theorem are identical (say, e.g., $p_1 = p_2$), then the assertion surely holds true (since then, in the example case, $p_1\mathrm{B}p_2p_3$). Hence we assume that all the points differ from each other. Let p_4 and p_5 be the endpoints of the segment s; hence $s = p_4p_5$. By *GEO 5* $s = p_4p_1 + p_1p_5$ with $p_1 = p_4p_1 \cdot p_1p_5$. Assume that p_2 and p_3 belong to different component segments of s, e.g.: $p_2\mathrm{B}p_4p_1$ and $p_3\mathrm{B}p_1p_5$ (the converse distribution is treated in analogous way). Two further applications of *The5* yield $p_4p_1 = p_4p_2 + p_2p_1$ and $p_1p_5 = p_1p_3 + p_3p_5$ where p_2 and p_3 are, respectively, the only common points of the component segments. Hence both $p_1 \notin \mathrm{P}^{>}p_4p_2$ and $p_1 \notin \mathrm{P}^{>}p_3p_5$. Again by *GEO 5* we have $s = p_4p_2 + p_2p_5$ with $p_2 = p_4p_2 \cdot p_2p_5$. From the latter and $p_1 \notin \mathrm{P}^{>}p_4p_2$ we conclude $p_1\mathrm{B}p_2p_5$ and, since $p_3\mathrm{B}p_1p_5$, $p_3\mathrm{B}p_2p_5$, too. A further application of *The 5* yields $p_2p_5 = p_2p_3 + p_3p_5$ with $p_3 = p_2p_3 \cdot p_3p_5$. We already know that $p_1 \notin \mathrm{P}^{>}p_3p_5$ and thus infer $p_1\mathrm{B}p_2p_3$. — Now assume that p_2, p_3 belong to the same component segment of our first division of s into p_4p_1 and p_1p_5. Assume $p_2, p_3 \in \mathrm{P}^{>}p_4p_1$; the remaining possibility is treated in an analogous way. *The 5* yields that $p_4p_1 = p_4p_2 + p_2p_1$ with $p_2 = p_4p_2 \cdot p_2p_1$. Hence either $p_3\mathrm{B}p_1p_2$ and nothing remains to prove, or $p_3 \in \mathrm{P}^{>}p_4p_2$. In the latter case we know by *The 20* that $p_2 \notin \mathrm{P}^{>}p_4p_3$. But by a final application of *The 5* we have $p_4p_1 = p_4p_3 + p_3p_1$ with $p_3 = p_4p_3 \cdot p_3p_1$; therefore $p_2\mathrm{B}p_1p_3$. ■

Def 15 does not require that the two boundary points delimiting the position of the third point differ. If they do not, the point in between them is identical to them.

The 23: $p_1\mathrm{B}p_2p_2 \rightarrow p_1 = p_2$

This is immediate from the idempotent law for segments *The 16*. *The 23* is an axiom of Tarski's system of Euclidean geometry presented and investigated in [50]. There it is called the *identity axiom for the betweenness relation*. Pasch [51], who is celebrated for his analysis of the order relation, postulated that points connected by the B-relation differ from each other, and Hilbert [12] followed him in this. The B-relation assumed here may be easily modified in the way suggested by the Pasch-Hilbert-view.

Def 16: (a) $\mathrm{B}^+ \underset{\mathrm{def}}{=\!=} \mathrm{B} \cap \{\langle p_1, p_2, p_3\rangle \mid \bigwedge_{1\leq i<j\leq 3} p_i \neq p_j\}$
(b) $p_2\mathrm{B}^+p_1p_3 \underset{\mathrm{def}}{\Longleftrightarrow} \langle p_1, p_2, p_3\rangle \in \mathrm{B}^+$

For the concept of a segment and for betweenness-relation, which are both basic in his axiomatisation of geometry, Pasch [51, §1] postulates nine axioms. In the present framework they can be formulated as shown in Tab. I. Of these axioms, I. and IV. are just a special cases of *The 7* and the "monotone law" *Theorem 14*-(b), respectively. Furthermore, V. easily follows from *GEO 5*. VI. says that a segment p_1p_2 is always extendable beyond its boundary point p_2. We adopt it as a basic principle of our system.

GEO 7: $\exists p_3.p_2\mathrm{B}^+p_1p_3$

IX. is a *dimensionality axiom*. Since we wish to remain neutral here with respect to dimensionality, we do not accept this axiom.

I.	$\overset{1}{\exists} s.s = p_1p_2$
II.	$\exists p_2.p_2\mathrm{B}^+p_1p_3$
III.	$p_2\mathrm{B}^+p_1p_3 \rightarrow \neg p_1\mathrm{B}^+p_2p_3$
IV.	$p_2\mathrm{B}^+p_1p_3 \rightarrow p_1p_2 \in \mathrm{P}^{>}p_1p_3$
V.	$p_2\mathrm{B}^+p_1p_3 \wedge p_4 \notin \mathrm{P}^{>}p_1p_2 \cup \mathrm{P}^{>}p_2p_3 \rightarrow p_4 \notin \mathrm{P}p_1p_3$
VI.	$\exists p_3.p_2\mathrm{B}^+p_1p_3$
VII.	$p_2\mathrm{B}^+p_1p_3 \wedge p_2\mathrm{B}^+p_1p_4 \rightarrow p_3\mathrm{B}^+p_1p_4 \vee p_4\mathrm{B}^+p_1p_3$
VIII.	$p_2\mathrm{B}^+p_1p_3 \wedge p_1\mathrm{B}^+p_2p_4 \rightarrow p_1\mathrm{B}^+p_3p_4$
IX.	$\exists p_3.[\neg p_1\mathrm{B}^+p_2p_3 \wedge \neg p_2\mathrm{B}^+p_1p_3 \wedge \neg p_3\mathrm{B}^+p_1p_2]$

TABLE I
PASCH'S AXIOMS FOR SEGMENTS AND BETWEENNESS

Pasch's axiom II. is accepted here in the slightly modified but equivalent form *GEO 8*.

GEO 8: $p_1 \neq p_2 \rightarrow p_1p_2 \neq p_1 + p_2$

The 8 requires each non-punctual segment to contain at least two points, namely its boundaries. *GEO 8* excludes "hollow" segments just consisting of their boundaries. It thus says that the relations B and B^+ are *dense*. Hence it may be called the *denseness* axiom; cf. [1, p. 146].[22] It corresponds to the second of Hilbert's "axioms of order"; cf. [12, chap. I, §3].[23]

Within our framework, then, we can prove Pasch's VIII. by means of the principles of decomposability, ramification and denseness. Actually VIII. refers to a special constellation considered in the straightness axiom *GEO 6*. If in that axiom $s_1 = p_3p_1$ and $s_2 = p_2p_4$, then we expect that $s_1 + s_2 = p_3p_4$; cf. the left hand side of Fig. 4.

The 24: $p_1 \neq p_2 \wedge p_2\mathrm{B}p_3p_1 \wedge p_1\mathrm{B}p_2p_4 \rightarrow$
$p_3p_1 + p_2p_4 = p_3p_4$

Proof: According to *GEO 6*, $p_3p_1 + p_2p_4$ is a segment, and according to *The 8* and *The 15* it contains the segment p_3p_4 as a part. Thus it remains to be shown that also conversely $p_3p_1 + p_2p_4 \in \mathrm{P}^{>}p_3p_4$ (∗). — In order to prove this, we first deal with some special cases. (a) If $p_1 = p_3$, then $p_1 = p_3p_1 = p_3$. But this would, in contradiction to the assumption of the theorem, imply that $p_1 = p_2$ since $p_2\mathrm{B}p_3p_1$. Hence $p_1 \neq p_3$. — (b) Furthermore, $p_2 \neq p_4$, too. For otherwise we would have $p_2 = p_2p_4 = p_4$ and hence from the hypothesis $p_1\mathrm{B}p_2p_4$ $p_1 = p_2$, which again contradicts the assumption $p_1 \neq p_2$. — (c) Finally, $p_3 \neq p_4$, too. For otherwise we had $p_1\mathrm{B}p_2p_3$ because of $p_1\mathrm{B}p_2p_4$. But since $p_2\mathrm{B}p_3p_1$, $p_3p_1 = p_3p_2 + p_2p_1$ with

[22] In [1], that axiom is formulated for a segment operation which maps pairs of points p_1 and p_2 to the *open* segment bounded by those two points; cf. Fn. 3. This means that the points do not belong to the segment which they delimit.

[23] Hilbert, however, conceives of order as a relation restricted to the points of some given line. We shall return to the topic of lines at the end of the present subsection.

$p_2 = p_3p_2 \cdot p_2p_1$. As we have just seen, however, $p_1\mathrm{B}p_2p_3$; this would yield $p_1 = p_2$, once more in contradiction to the assumption. (d) We may assume that $p_4 \notin \mathrm{P}^{>}p_3p_1$ (and so particularly $p_1 \neq p_4$). Otherwise, by decomposition, $p_4\mathrm{B}p_3p_2$ (d.1) or $p_4\mathrm{B}p_2p_1$ (d.2). The latter would imply $p_1 = p_4$ by *The 20*. Since the assumption $p_2\mathrm{B}p_3p_1$ implies $p_2p_4 \in \mathrm{P}^{>}p_3p_1$ by *The 15*, we had $p_3p_1 + p_2p_4 = p_3p_1$ which together with $p_1 = p_4$ implies $(*)$, which concludes the case (d.2). — Assume then (d.1). By decomposition $p_3p_4 + p_4p_2 = p_3p_2$ with $p_4 = p_3p_4 \cdot p_4p_2$. From this and from (b) above we conclude that $p_2 \notin \mathrm{P}^{>}p_3p_4$. Again by decomposition $p_3p_4 + p_4p_1 = p_3p_1$; thus $p_2\mathrm{B}p_4p_1$. But then with the assumption $p_1\mathrm{B}p_2p_4$ and *The 20* $p_1 = p_2$ in contradiction to the theorem's assumption. — (e) Because of a similar reasoning we may also assume that $p_3 \notin \mathrm{P}^{>}p_2p_4$ (and thus in particular $p_2 \neq p_3$). — In order to prove $(*)$, it is sufficient to show that $p_1, p_2 \in \mathrm{P}^{>}p_3p_4$. From the latter and *The 15* it follows that p_3p_1 and p_2p_4 are both parts of p_3p_4 which implies $(*)$ by *The 3*. — Assume first that p_1 were not a part of p_3p_4. We may take it for granted that $p_4 \notin \mathrm{P}^{>}p_3p_1$; cf. (d) above. Hence we may apply *The 21* and conclude that $p_3 = p_3p_4 \cdot p_3p_1$ (f). Now consider p_2p_4. From (f), the assumption $p_2\mathrm{B}p_3p_1$, and the fact that—according to (b), (c) above— p_2 differs from both p_3 and p_4, we infer that $p_2 \notin \mathrm{P}^{>}p_3p_4$. Furthermore we may assume that $p_3 \notin \mathrm{P}^{>}p_2p_4$; cf. (e) above. Thus we may once again by *The 21* conclude that $p_4 = p_3p_4 \cdot p_2p_4$ (g). By (f) and (g), we have that p_3 and p_4 are the only points shared by p_3p_4 and $p_3p_1 + p_2p_4$. We had already proven that $p_3p_4\mathrm{P}^{>}p_3p_1 + p_2p_4$ and hence $p_3p_4(p_3p_1+p_2p_4) = p_3p_4$. But since the factors of that product share only the two points mentioned, we are forced to conclude that $p_3p_4 = p_3 + p_4$ which contradicts denseness. — The same result is reached by a parallel argumentation which starts from the assumption that $p_2 \notin \mathrm{P}^{>}p_3p_4$.

Pasch's VII. is a consequence of the ramification property; cf. *The 21*.

The 25: $\quad p_2\mathrm{B}^{+}p_1p_3 \wedge p_2\mathrm{B}^{+}p_1p_4 \rightarrow p_3\mathrm{B}^{+}p_1p_4 \vee p_4\mathrm{B}^{+}p_1p_3$

Proof: Since we are concerned with strict betweenness here, $p_1 \neq p_2$. Hence the two segments p_1p_3 and p_1p_4 share more than just one point; thus $p_1 \neq \inf(p_1p_3, p_1p_4)$. By the ramification property then: $\neg[p_4 \notin \mathrm{P}^{>}p_1p_3 \wedge p_3 \notin p_1p_4]$, i.e., $p_3 \in \mathrm{P}^{>}p_1p_4 \vee p_4 \in \mathrm{P}^{>}p_1p_3$, hence (since, by hypothesis, both $p_1 \neq p_3$ and $p_1 \neq p_4$): $p_3\mathrm{B}^{+}p_1p_4 \vee p_4\mathrm{B}^{+}p_1p_3$. ■

We have thus proven (or, in the case of II. and VI. simply taken over) all of Pasch's axioms for lines. The celebrated axiom called after him ("Pasch's Axiom"; Hilbert's fourth "axiom of order"), however, is not included within the list of Tab. I because it makes its appearance in [51] only in the book's second paragraph, which deals with planes. We shall return to Pasch's Axiom in the next section. We conclude the present section by a brief consideration of (unbounded) lines (as opposed to bounded segments).

Pasch [51, p. 4] rejects the notion of an infinitely extended line since it does not "correspond to anything perceivable". Nevertheless, he introduces lines into his system by a special procedure which he calls "implicit definition" (and is to be distinguished from "definition by axioms" also called thus). The procedure is more closely described in [52], where Pasch relates it to the doctrine of the As-If of the neo-Kantian philosopher Hans Vaihinger. It essentially consists in the introduction of a kind of new, "fictitious" objects which, however, can be uniquely characterized by their relationships to "real" objects. In our mereological framework we construct lines as sums of certain points. The following definition mirrors the set-theoretical procedure of [53, p. 50].

Def 17: $\quad \mathrm{L} \underset{\text{def}}{=\!=} \{\langle u_1, p_1, p_2\rangle \mid p_1 \neq p_2 \wedge u_1 = \sup(p_3 \mid p_1\mathrm{B}p_3p_2 \vee p_3\mathrm{B}p_1p_2 \vee p_2\mathrm{B}p_1p_3)\}$

Obviously, L is a (partial) operation. Coppel [53, p. 49–52] develops a theory of lines based on four axioms (L1)–(L4). (L1) is the idempotent law *The 16*. Coppel's (L2) says that $p_2\mathrm{B}p_1p_4$ if $p_2\mathrm{B}p_1p_3$, $p_3\mathrm{B}p_2p_4$, and $p_2 \neq p_3$. This is an immediate consequence of our *The 24*. (L3) is the ramification principle; cf. *The 21*. Finally (L4) is that part of *GEO 5* which is concerned with the supremum-operation (omitting the part postulating that the combined segments share a single point). Since all of Coppel's axiom are provable in our framework (and his proofs do not make use of set-theoretical procedures not available in our class-theoretical framework), the theorems proven by him can directly be taken over into our framework. This includes his result that a line is uniquely determined by any two points lying on it and that the points on a line can be arranged in a unique way in a total linear order. Because of *GEO 8*, we may add that this order is dense.

C. The Axioms of Peano and Pasch

The structure of points and lines described at the end of the previous section is known as a "line space"; cf. [1, p. 155]. In a line space two distinct points belong to a unique line, each line has at least two points, and the points on each line are arranged in a total linear dense order. The only condition entering into the definition of a line space which has not been established yet is the so-called *Peano Axiom* (cf. the left-hand side of Fig. 6). However, this axiom can be proven in our framework.

The 26: $\quad p_4\mathrm{B}p_2p_3 \wedge p_5\mathrm{B}p_1p_4 \rightarrow \exists p_6 \in \mathrm{P}^{>}p_1p_2.p_5\mathrm{B}p_3p_6$

Proof: Suppose that both $p_4\mathrm{B}p_2p_3$ and $p_5\mathrm{B}p_1p_4$. Then p_5 belongs to $[p_1, p_2, p_3]$ which, by *GEO 4*, is $\sup(s \mid \exists p_4 \in \mathrm{P}^{>}p_2p_3.s = p_1p_4)$. By the same axiom, however, $[p_1, p_2, p_3] = \sup(s \mid \exists p_6 \in \mathrm{P}^{>}p_1p_2.s = p_3p_6)$. Hence $p_5 \in \mathrm{P}^{>}p_3p_6$ for some $p_6 \in \mathrm{P}^{>}p_1p_2$. ■

From the Peano Axiom we now deduce Pasch's Axiom. Pasch himself formulates his axiom [51, p. 20] as stating a relationship between a triangle and a segment entering the inner of the triangle by crossing one of its edges. The theorem then says that either the entering segment itself or a prolongation of it leaves the triangle by crossing (either a vertex of the triangle or) exactly one of the two other edges. Here we prove another formulation. Pasch's own version of

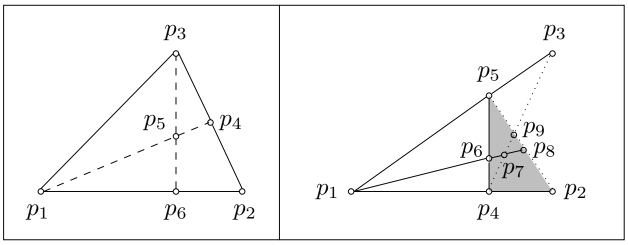

Fig. 6. The Peano Axiom and Proof of Pasch's Axiom

his axiom, however, is provable from the version used here, cf. [53, p. 57f]. Conversely, our version given below is implied by Pasch's theorem; cf. [51, p. 25].

The 27: $p_4 \mathrm{B} p_1 p_2 \wedge p_5 \mathrm{B} p_1 p_3 \rightarrow \langle p_2 p_5, p_3 p_4 \rangle \in \mathrm{O}$

Proof: We[24] may presuppose that none of the three points p_1, p_2, p_3 lies on the segment determined by the two others. For in that case all the points involved lie on that segment. In this case the theorem is easily proven by checking a number of trivial possibilities. By *GEO 8* there exists a point p_6 between p_5 and p_4. Obviously, $p_1 \neq p_6$ since otherwise p_1, p_2, p_3 would lie on a common segment contrary to our initial assumption. For p_6 two applications of *The 26* yield points p_7 and p_8 on, respectively, $p_3 p_4$ and $p_2 p_5$ such that both $p_6 \mathrm{B} p_1 p_7$ and $p_6 \mathrm{B} p_1 p_8$. By *GEO 6* p_1, p_6, p_7, and p_8 lie on a segment; hence, by *The 22* one of the three points p_6, p_7, and p_8 must lie within the segment spanned by the two others. If $p_6 \mathrm{B} p_7 p_8$, it would follow that all three points were identical and then that point would be a common point of $p_2 p_5$ and $p_3 p_4$. Hence $p_7 \mathrm{B} p_6 p_8$ or $p_8 \mathrm{B} p_6 p_7$. The first case is that displayed in the left hand side of Fig. 6. We consider only this case; the argument for the other case is completely analogous. The segment $p_6 p_8$ is a part of the triangle $\triangle p_4 p_5 p_2$ (shaded in Fig. 6); therefore the point p_7 also belongs to that triangle. It follows by an application of *GEO 4* that $p_7 \mathrm{B} p_4 p_9$ for some p_9 with $p_9 \mathrm{B} p_2 p_5$. Applying now *The 26* to $\triangle p_1 p_2 p_3$ and the point p_9 lying on the segment $p_2 p_5$ we infer that there must be a point p_{10} between p_1 and p_2 such that $p_9 \mathrm{B} p_3 p_{10}$. Furthermore, $p_9 \neq p_3$, for else p_1, p_2, p_3 would again lie on a single segment which is impossible according to our initial assumption. The segments $p_3 p_{10}$ and $p_3 p_4$ share thus two points, namely p_3 and p_9. Hence, by *The 6*, the sum s is a segment. This segment s contains both p_4 and p_{10}. If these were two different points, then, again by *The 6*, the sum of s with $p_1 p_2$ would be a segment containing all three of p_1, p_2, p_3. Hence we conclude that $p_4 = p_{10}$. But then $p_9 \mathrm{B} p_3 p_4$ and $p_3 p_4$ and $p_2 p_5$ overlap in p_9. ■

We conclude the presentation of our theory of space by some remarks relating to two topics addressed in the Introduction (sec. I), namely "coordinates" and the idea of a "geometric algebra".

[24] Van de Vel [1, p. 144] gives another proof of Pasch's Axiom making use only of the Ramification Principle instead of the stronger Straightness Principle. The proof given here employs the same idea as is used by Coppel [53, p. 86] but applies *GEO 4* in order to infer the existence of point p_9.

V. Coordinates

The first step in turning a qualitative theory of space into a "quantitative" one by the introduction of coordinates is to define operations of addition and multiplication for the segments of a line (or, when a certain point of the line is distinguished as the "origin" or "zero point", for the points of that line) by help of Desargue's theorem; cf., e.g., [12, § 24] or [50, Part I, § 14]. Now Coppel [53, p. 54f] defines a "linear geometry" as a structure fulfilling (set-theoretic counterparts) of Pasch's principle VIII. (which follows from our *The 24*),[25] our *The 21* (the "ramification principle"), our *GEO 5* (the "decompososability principle") and the Peano Axiom (our *The 26*). Furthermore, he shows [53, pp. 125–131] that in each dense linear geometry (thus in each linear geometry in which our *GEO 8* is valid) of dimensionality greater than 2, Desargue's theorem holds true in the following form: Let $\mathrm{L}(p_1 p_2)$, $\mathrm{L}(p_3 p_4)$, $\mathrm{L}(p_5 p_6)$ three distinct lines with a common point p which is different from p_j $(1 \leq j \leq 6)$. If corresponding sides of the two triangles $\triangle p_1 p_3 p_4$ and $\triangle p_2 p_4 p_6$ intersect, then the three points of intersection lie on the same line. Finally, it is shown by Coppel [53, ch. 7] that suitable additive and multiplicative operations for points can be defined by use of Desargue's theorem in such a way that a linear geometry in which this theorem is valid may be embedded into (the projective completion of) an ordered skew field. Thus in order to introduce coordinates we can either postulate that our space is three-dimensional or we may directly require that Desargue's theorem is true.

Of course, if it is desired that the order of points on a line have Dedekind's cut property something more is necessary. In the common axiomatisation of geometry the axiom of continuity guaranteeing the cut property makes use of quantification over sets of points which is not available in LC. However, since regions are just the sums of their points, first-order quantification over regions mimics quantification over point sets. Thus it would be interesting to see how far one gets with something like $\exists p.\forall p_1 \in \mathrm{P}^{>} u_1, p_2 \mathrm{P}^{>} u_2 . p_1 \mathrm{B} p p_2 \rightarrow \exists p_0 \forall p_1 \in \mathrm{P}^{>} u_1, p_2 \mathrm{P}^{>} u_2 . p \mathrm{B} p_1 p_2$ which results from Tarski's axiom on continuity [50, p. 14] by replacing variables for point sets by variables for regions.

VI. Note on the idea of a "geometric algebra"

In the Introduction (sec. I) we considered Leibniz' idea of a geometric algebra in which one can directly calculate with points and lines without encoding these geometric entities by (pairs and sets of) numbers. Furthermore, the geometric system presented here has been inspired in many respects by Prenowitz' "join geometry" which shares the "algebraic vision" with Leibniz' idea of a *characteristica geometrica*. However, that system with its axioms involving logical connectives and quantifiers is more in line with the ancient Euclidean procedure than with modern algebraic theories —

[25] Actually, not VIII. itself but an equivalent "mirror image" of it in which the sequence of points concerned is inverted is used by Coppel.

like, e.g., ring theory or lattice theory — which fix their models by lists of equations.

Nevertheless there is the possibility to develop within our framework parts of geometry in an "equational manner". In the present section we briefly describe an example for this. In the Introduction (sec. I) it was pointed out that the "associative law" $p_1(p_2p_3) = (p_1p_2)p_3$ for Prenowitz' join operation is only interpretable on the basis of certain conventions. The problem is that the join operation is defined for points while, one subterm (namely, respectively "(p_2p_3)" and "(p_1p_3)") of the main terms on both side of the equation stating the law refers to a segment rather than to a point. In the framework presented in this article this is not really a problem. We have just to replace *Def 14*-(c) by the two definitions $p_1u_1 \underset{\text{def}}{=\!=} u_1p_1 \underset{\text{def}}{=\!=} [p_1, u_1]$. Since the universal variable "u_1" is substitutable by variables of both sort **p** and sort **s**, this simultaneously defines the operations of (1) joining points, (2) joining a point to a segment, and (3) joining a segment to a point; and their would be no problem with a terms like "$p_1(p_2p_3)$" and "$(p_1p_2)p_3$" entering into the formulation of the associative law.

However, this solution suffers from two shortcomings. (1st) It seems too restrictive by requiring that one argument of the generalized join operation is still a point. Should one not define the operation in a completely general way admitting for arguments of any sort? (2nd) It does not match Prenowitz intention that the join of a point and a segment is built up by the segments joining the point argument and the individual points of the segment argument. Thus it would be more adequate to keep the old *Def 14*-(c) for joining points and to supplement it by the following general one.[26]

$$\textit{Def 18: } u_1 \circ u_2 \underset{\text{def}}{=\!=} \sup(s \mid \exists p_1 \in \mathrm{P}^{>}u_1, p_2 \in \mathrm{P}^{>}u_2.s = p_1p_2)$$

Obviously, $p_1 \circ p_2 = p_1p_2$; we do not therefore make a notational distinction between the two operations. It is immediate that the general join operation is commutative. Furthermore, it is obvious that is idempotent for convex regions. Thus two of the algebraic laws valid for his special join operation hold true for the more general one when it is restricted to convex regions.

The 28: Idempotency $cc = c$
Commutativity $c_1c_2 = c_2c_1$

Is the third law postulated to hold by Prenowitz for his join operation, thus assiociativity, valid (for convex regions), too? In order to show that $c_1(c_2c_3) = (c_1c_2)c_3$, it suffices in view of *The 6* to show that $c_1(c_2c_3)$ and $(c_1c_2)c_3$ have exactly the same points. Let us consider a point p_4 of $c_1(c_2c_3)$; we have to show that $p_4 \in \mathrm{P}^{>}(c_1c_2)c_3$. From $p_4\mathrm{P}^{>}c_1(c_2c_3)$ we conclude the existence of points $p_j \in \mathrm{P}^{>}c_j$ $(1 \leq j \leq 3)$ and of a point $p_5\mathrm{B}p_2p_3$ such that $p_4\mathrm{B}p_1p_5$; cf. the constellation built up by the solids line in the left hand diagram of Fig. 7. That p_4

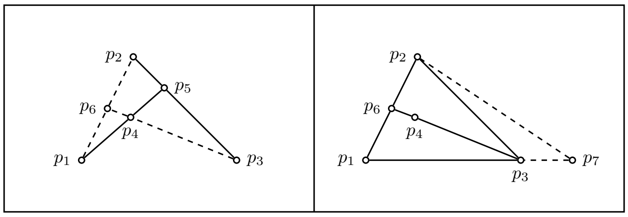

Fig. 7. Associativity of the generalized join operation

belongs to $(c_1c_2)c_3$ means that there are points $p'_j \in \mathrm{P}^{>}c_j$ $1 \leq j \leq 3$ and a point p'_5 such that $p'_5\mathrm{B}p'_1p'_2$ and $p_4\mathrm{B}p'_5p'_3$. Considering the situation depicted on the left hand side of Fig. 7, it suggests itself to set $p'_j := p_j$ $1 \leq 3$ and to chose p'_5 as the point of intersection p_6 of p_1p_2 and the line through the segment p_3p_4; cf. the constellation of dashed lines in the left part of Fig. 7.

What then remains to be shown is that p_6 really exists, i.e., that the segment p_3p_4 can be extended so that it intersects with p_1p_2. But one readily recognizes that the situation described is just a "Pasch-configuration". Using *GEO 7*, we may extend p_1p_3 to a point p_7 such that $p_3\mathrm{B}p_1p_2$. Then the line through p_3p_4 is a line entering the triangle $\triangle p_1p_7p_2$ through its edge p_1p_7. The original version of Pasch's Axiom requires that this line leaves the triangle through one of the two other edges. In our case p_6 must be incident with p_1p_2. Assume that p_6 were a part of p_7p_2. The only point that p_7p_2 has in common with $\triangle p_1p_3p_2$ is p_2;[27] hence we had $p_2 = p_6$ and thus nevertheless $p_6 \in p_1p_2$. So we have proven that $p_4 \in \mathrm{P}^{>}(p_1p_2)p_3$ and by this that $\mathrm{P}^{>}p_1(p_2p_3) \subseteq \mathrm{P}^{>}(p_1p_2)p_3$. The converse of this can be shown by an analogous argument. Summing up, we have thus proven associativity.

The 29: Associativity $c_1(c_2c_3) = (c_1c_2)c_3$

Prenowitz' [9, p. 55] three basic algebraic laws (J2), (J3), and (J4) of his join geometry are (*modulo* the difference explained in Fn. 3 above) just special cases of the more general principles given here and can be derived from these principles because of *MER 2*.

VII. Conclusion

We have provided a theory of space formulated in a mereological framework which is based on the notion of convexity. Using mereological concepts, segments of straight lines have been explained as the convex hulls of the sum of two points. What "straightness" exactly means for thus defined segments has been determined by two axioms. Two further axioms have been included into the system in order to describe the linear arrangement of the points of a segment. Finally, it has been illustrated by examples how a more algebraic approach to geometry, envisaged already by Leibniz, can be developed within the framework presented here.

[26] *Def 18* corresponds in our mereological framework to a similar definition provided by Prenowitz and Jantocziak in their set-theoretic setting; cf. [9, p. 55].

[27] In order to see this, one has to use *GEO 6*.

Acknowledgment

The authors would like to thank Carola Eschenbach, Christopher Habel, Felix Lindner, and Johanna Seibt for a series of very fruitful discussions in the Danish-German research initiative *Making Space*.

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