Computing the minimal solutions of finite fuzzy relation equations on linear carriers

Juan Carlos Díaz-Moreno, Jesús Medina, Esko Turunen

Abstract—Fuzzy relation equations is an important tool for managing and modeling uncertain or imprecise datasets, which has useful applied to, e.g. approximate reasoning, time series forecast, decision making, fuzzy control, etc. This paper considers a general fuzzy relation equation, which has minimal solutions, if it is solvable. In this case, an algebraic characterization is introduced which provides an interesting method to compute minimal solutions in this general setting.

I. INTRODUCTION

Fuzzy relation equations were introduced by E. Sanchez in the seventies [11]. These equations have widely been studied in different papers [1], [3], [6]. For example, they have proven that the set of solutions of solvable fuzzy relation equations is a upper-preserving complete lattice in which the greatest solutions is completely determined. Nevertheless, the computation of minimal solutions is not so direct. These solutions have been studied in several papers [2], [4], [10], [12], [17], [15], [16], [18], [13] and several algorithms have been developed, but in restrictive frameworks, restrictions that limit the flexibility of the possible applications.

Hence, first of all, it is fundamental to study general frameworks in which the minimal solutions of each solvable fuzzy relation equation exist and that each solution will be between the greatest solution and a minimal solution.

This paper considers a general setting, in which the operators may neither be commutative nor associative and they only need to be monotone and residuated inf-preserving mappings of non-empty sets on the right argument. The linearity of the carrier, together with the inf-preserving property, ensures the existence of minimal solutions whenever a solution exists.

Mainly, this paper introduces a procedure in order to obtain the minimal solutions of a solvable of the introduced general fuzzy relation equations. Moreover, we have presented a detailed algorithm to compute these important solutions, together with several illustrative examples.

II. GENERAL FUZZY RELATION EQUATIONS

Throughout this paper we will consider a complete linear lattice \((L, \preceq)\), in which the bottom and the top elements exist and they are denoted as 0, 1, respectively. Given a set \(V\), the ordering \(\preceq\) in the lattice induces a partial order on the set of \(L\)-fuzzy subsets of \(V, L^V\). This ordering provides to \(L^V\) the structure of a complete lattice.

A general residuated operator will also be used in this paper to define the fuzzy relation equation, as in [8]. This residuated operator will be denoted as \(\odot : L \times L \rightarrow L\), which is order preserving in both arguments and there exists another operator \(\rightarrow : L \times L \rightarrow L\), satisfying the following adjoint property with the conjunctor \(\odot\)

\[
x \odot y \preceq z \quad \text{if and only if} \quad y \preceq x \rightarrow z
\]

for each \(x, y, z \in L\). This property is equivalent to say that \(\odot\) preserves suprema in the second argument; \(x \odot \bigvee\{y \mid y \in Y\} = \bigvee\{x \odot y \mid y \in Y\}\), for all \(Y \subseteq L\).

These operators, as were noted in [8], generalize other kind of residuated pairs [7], [5], since only the monotonicity and the adjoint property are considered.

**Definition 1.** Given the pair \((\odot, \rightarrow)\), a fuzzy relation equation is the equation:

\[
R \odot X = T,
\]

where \(R : U \times V \rightarrow L, T : U \times W \rightarrow L\) are given finite \(L\)-fuzzy relations and \(X : V \times W \rightarrow L\) is unknown; and \(R \odot X : U \times W \rightarrow L\) is defined, for each \(u \in U, w \in W\), as

\[
(R \odot X)(u, w) = \bigvee\{R(u, v) \odot X(v, w) \mid v \in V\}.
\]

It is well known that the fuzzy relation equation (2) has a solution if and only if

\[
(R \Rightarrow T)(v, w) = \bigwedge\{R(u, v) \rightarrow T(u, w) \mid u \in U\}
\]

is a solution and, in that case, it is the greatest solution, see [7], [11], [14].

III. COMPUTING MINIMAL SOLUTIONS ON LINEAR LATTICES

**Definition 2.** Given an operator \(\odot : L \times L \rightarrow L\), we will say that it holds the IPNE-condition (making reference to that \(\odot\) is Infimum Preserving of arbitrary Non-Empty sets), if it verify

\[
a \odot \bigwedge B = \bigwedge\{a \odot b \mid b \in B\}
\]

for each element \(a \in L\) and each non-empty subset \(B \subseteq L\).
From now on, let us consider a general solvable fuzzy relation equation (2), where \( R, X, T \) are finite and \( \circ \) satisfies the IPNE-condition.

First of all, the auxiliary sets \( V_{uw} \) need to be introduced, which are associated with the elements \( u \in U, w \in W \) and the greatest solution \( R \Rightarrow T \). Since for each \( u \in U, w \in W \)
\[
\sqrt{(R(u,v) \circ (R \Rightarrow T)(v,w) | v \in V)} = T(u,w),
\]
\( L \) is linear and \( V \) is finite, there exists at least one \( v_s \in V \) validating the equation
\[
R(u,v_s) \circ (R \Rightarrow T)(v_s,w) = T(u,w).
\]
Therefore, the set
\[
V_{uw} = \{ v \in V | R(u,v) \circ (R \Rightarrow T)(v,w) = T(u,w) \}
\]
is not empty and, for all \( v \notin V_{uw} \), the strict inequality
\[
R(u,v) \circ (R \Rightarrow T)(v,w) < T(u,w)
\]
holds.

Each \( v_s \) in \( V_{uw} \) will provide a fuzzy subset \( S_{uw} \) as follows: Given \( v_s \in V_{uw} \), we have that
\[
\{ d \in L | R(u,v_s) \circ d = T(u,w) \} \neq \emptyset
\]
and the infimum \( \land \{ d \in L | R(u,v_s) \circ d = T(u,w) \} = e_s \)
also satisfies the equality
\[
R(u,v_s) \circ e_s = T(u,w)
\]
by the IPNE-condition. These elements are used to define the fuzzy subsets of \( V, Z_{uw} : V \rightarrow L \), defined by
\[
Z_{uw}(v) = \begin{cases} e_s & \text{if } v = v_s \\ 0 & \text{otherwise} \end{cases}
\]
which form the set \( Z_{uw} \), that is \( Z_{uw} = \{ Z_{uw} | v_s \in V_{uw} \} \), for each \( u \in U, w \in W \). These sets will be used to characterize the set of solutions of Equation (2) by the notion of covering.

**Theorem 3.** The L-fuzzy relation \( X : V \times W \rightarrow L \) is a solution of a solvable equation (2) if and only if \( X \preceq (R \Rightarrow T) \) and, for each \( w \in W \), the fuzzy subset \( X_w : V \rightarrow L \), defined by \( X_w(v) = X(v,w) \), is a cover of \( \{ Z_{uw} | u \in U \} \).

As a consequence, the minimal solutions are characterized by the minimal covers.

**Corollary 4.** \( X : V \times W \rightarrow L \) is a minimal solution of Equation (2) if and only if, for each \( w \in W \), \( X_w : V \rightarrow L \), defined by \( X_w(v) = X(v,w) \), is a minimal cover of \( \{ Z_{uw} | u \in U \} \).

Hence, from the corollary above, minimal solutions of the fuzzy relation equation (2) are obtained from \( R \Rightarrow T \). Next, the detailed algorithms are introduced.

Module MINIMAL_COVERING uses an usual algorithm in order to compute minimal covering of subsets.

**Example III.1.** Let us assume the standard MV–algebra [9], that is, \( L = [0,1] \) is the unit interval, \( \circ : L \times L \rightarrow L \) is the Lukasiewicz operator defined by \( x \circ y = \max(0, x + y - 1) \) and \( \rightarrow : L \times L \rightarrow L \) is its residuated implication, defined by \( y \rightarrow z = \min(1, 1 - y + z) \), for all \( x, y, z \in [0,1] \).

**Algorithm 1: PMINSOLUTIONS \((R, T)\)**

Given \( U = \{ u_1, u_2, u_3 \}, V = \{ v_1, v_2, v_3 \} \), \( W = \{ w_1, w_2, w_3 \} \) and the fuzzy relation equations, defined from the following tables

\[
\begin{array}{ccc}
R & v_1 & v_2 & v_3 \\
\hline
u_1 & 0.9 & 0.5 & 0.9 \\
u_2 & 0.2 & 0.9 & 0.7 \\
u_3 & 0.8 & 0.6 & 0.9 \\
\end{array}
\]

\[
\begin{array}{ccc}
T & w_1 & w_2 & w_3 \\
\hline
u_1 & 0.8 & 0.4 & 0.7 \\
u_2 & 0.6 & 0.7 & 0.3 \\
u_3 & 0.8 & 0.4 & 0.6 \\
\end{array}
\]

Direct computation shows that the relation \( R \Rightarrow T \), defined from the table

\[
\begin{array}{ccc}
R \Rightarrow T & w_1 & w_2 & w_3 \\
\hline
v_1 & 0.9 & 0.5 & 0.8 \\
v_2 & 0.7 & 0.8 & 0.4 \\
v_3 & 0.9 & 0.5 & 0.6 \\
\end{array}
\]

is the greatest solution of Equation (2). During the verification we go through the following calculations:

When computing \((R \circ (R \Rightarrow T))(u_1, w_1) = 0.8\), we consider the maximum of

\[
R(u_1, v_1) \circ (R \Rightarrow T)(v_1, w_1) = 0.9 \rightarrow (0.9 + 0.9 - 1) = 0.8 \\
R(u_1, v_2) \circ (R \Rightarrow T)(v_2, w_1) = 0.5 \rightarrow (0.5 + 0.7 - 1) = 0.2 \\
R(u_1, v_3) \circ (R \Rightarrow T)(v_3, w_1) = 0.9 \rightarrow (0.9 + 0.9 - 1) = 0.8
\]
Notice that from $v_1$ and $v_3$ we get the maximum. Hence, in order to obtain this maximum, we only need to consider \{v_1\} or \{v_3\}. Moreover, the values 0.9 associated with $v_1$ and 0.9 associated with $v_3$ cannot be decreased because, if we decrease them, a value less than 0.8 will be obtained in the computation and we do not reach a solution. Therefore, the first column of a solution of Equation (2) could be any column in the set:

$$Z_{1,1} = \{ \begin{pmatrix} 0.9 \\ 0 \\ 0.9 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0.9 \end{pmatrix} \}$$

However, we need to verify that the other two equalities also hold. Consequently, the equality $(R \circ (R \Rightarrow T))(u_2, w_1) = 0.6$ is studied similarly to the previous procedure. The value $(R \circ (R \Rightarrow T))(u_2, w_1)$ is the maximum of the values

$$R(u_2, v_1) \circ (R \Rightarrow T)(v_1, w_1) = 0.2 + 0.9 - 1 = 0.1$$
$$R(u_2, v_2) \circ (R \Rightarrow T)(v_2, w_1) = 0.9 + 0.7 - 1 = 0.6$$
$$R(u_2, v_3) \circ (R \Rightarrow T)(v_3, w_1) = 0.7 + 0.9 - 1 = 0.6$$

for which \{v_2\} or \{v_3\} is only necessary and so, the first column of a solution of Equation (2) could be one element of the set:

$$Z_{2,1} = \{ \begin{pmatrix} 0.7 \\ 0 \\ 0.9 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0.9 \end{pmatrix} \}$$

Finally, when computing $(R \circ (R \Rightarrow T))(u_3, w_1) = 0.8$ we pass by

$$R(u_3, v_1) \circ (R \Rightarrow T)(v_1, w_1) = 0.8 + 0.9 - 1 = 0.7$$
$$R(u_3, v_2) \circ (R \Rightarrow T)(v_2, w_1) = 0.6 + 0.7 - 1 = 0.3$$
$$R(u_3, v_3) \circ (R \Rightarrow T)(v_3, w_1) = 0.9 + 0.9 - 1 = 0.8$$

In this case, only $v_3$ is necessary and one column is only considered:

$$Z_{3,1} = \{ \begin{pmatrix} 0 \\ 0 \\ 0.9 \end{pmatrix} \}$$

We observe that

$$K = \begin{pmatrix} 0 \\ 0 \\ 0.9 \end{pmatrix} \in Z_{1,1} \cap Z_{2,1} \cap Z_{3,1},$$

so $K$ is the only minimal column which, in an intuitive sense, covers the set $Z_i = \{Z_{1,1}, Z_{2,1}, Z_{3,1}\}$. Moreover, we conclude that a fuzzy relation $X_1$, defined as

<table>
<thead>
<tr>
<th>$X_1$</th>
<th>$w_1$</th>
<th>$w_2$</th>
<th>$w_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_1$</td>
<td>0</td>
<td>0.5</td>
<td>0.8</td>
</tr>
<tr>
<td>$v_2$</td>
<td>0.8</td>
<td>0.4</td>
<td></td>
</tr>
<tr>
<td>$v_3$</td>
<td>0.9</td>
<td>0.5</td>
<td>0.6</td>
</tr>
</tbody>
</table>

solves the fuzzy relation equation (2).

Next, we consider the second column of $R \Rightarrow T$, which provides a different case. For $(R \circ (R \Rightarrow T))(u_1, u_2) = 0.4$ we have

$$R(u_1, v_1) \circ (R \Rightarrow T)(v_1, u_2) = 0.9 + 0.5 - 1 = 0.4$$
$$R(u_1, v_2) \circ (R \Rightarrow T)(v_2, u_2) = 0.5 + 0.8 - 1 = 0.3$$
$$R(u_1, v_3) \circ (R \Rightarrow T)(v_3, u_2) = 0.9 + 0.5 - 1 = 0.4$$

Hence, the maximum is obtained from $v_1$ or $v_3$ and, therefore, the following set is considered:

$$Z_{1,2} = \begin{pmatrix} 0.5 \\ 0 \\ 0.5 \end{pmatrix}$$

For $(R \circ (R \Rightarrow T))(u_2, w_2) = 0.7$ we have

$$R(u_2, v_1) \circ (R \Rightarrow T)(v_1, w_2) = 0$$
$$R(u_2, v_2) \circ (R \Rightarrow T)(v_2, w_2) = 0.9 + 0.8 - 1 = 0.7$$
$$R(u_2, v_3) \circ (R \Rightarrow T)(v_3, w_2) = 0.7 + 0.5 - 1 = 0.2$$

Consequently, the subset obtained is

$$Z_{2,2} = \begin{pmatrix} 0 \\ 0.8 \\ 0 \end{pmatrix}$$

For $(R \circ (R \Rightarrow T))(u_3, w_2) = 0.4$ we have

$$R(u_3, v_1) \circ (R \Rightarrow T)(v_1, w_2) = 0.8 + 0.5 - 1 = 0.3$$
$$R(u_3, v_2) \circ (R \Rightarrow T)(v_2, w_2) = 0.6 + 0.8 - 1 = 0.4$$
$$R(u_3, v_3) \circ (R \Rightarrow T)(v_3, w_2) = 0.9 + 0.5 - 1 = 0.4$$

Hence, the assumed subset of columns is

$$Z_{3,2} = \begin{pmatrix} 0 \\ 0.5 \\ 0 \end{pmatrix}$$

In this case, we observe that $Z_{1,2} \cap Z_{2,2} \cap Z_{3,2} = \emptyset$. However,

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0.8 \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0.8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \lor \begin{pmatrix} 0 \\ 0.8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \lor \begin{pmatrix} 0.8 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0 \end{pmatrix},$$

and $\begin{pmatrix} 0 \\ 0.8 \end{pmatrix}, \begin{pmatrix} 0.5 \\ 0 \end{pmatrix}$ are the only minimal columns which, again in an intuitive sense, cover the set $Z_2 = \{Z_{1,2}, Z_{2,2}, Z_{3,2}\}$. Moreover, we conclude that the fuzzy relations $X_2$ and $X_3$, defined as

<table>
<thead>
<tr>
<th>$X_2$</th>
<th>$w_1$</th>
<th>$w_2$</th>
<th>$w_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_1$</td>
<td>0</td>
<td>0</td>
<td>0.8</td>
</tr>
<tr>
<td>$v_2$</td>
<td>0</td>
<td>0.8</td>
<td>0.4</td>
</tr>
<tr>
<td>$v_3$</td>
<td>0.9</td>
<td>0.5</td>
<td>0.6</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$X_3$</th>
<th>$w_1$</th>
<th>$w_2$</th>
<th>$w_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_1$</td>
<td>0</td>
<td>0</td>
<td>0.8</td>
</tr>
<tr>
<td>$v_2$</td>
<td>0</td>
<td>0.8</td>
<td>0.4</td>
</tr>
<tr>
<td>$v_3$</td>
<td>0.9</td>
<td>0.5</td>
<td>0.6</td>
</tr>
</tbody>
</table>

solve the fuzzy relation (2). Finally, the values in the third column of $R \Rightarrow T$ are reduced.

For $(R \circ (R \Rightarrow T))(u_1, v_3) = 0.7$, we compute

$$R(u_1, v_1) \circ (R \Rightarrow T)(v_1, v_3) = 0.9 + 0.8 - 1 = 0.7$$
$$R(u_1, v_2) \circ (R \Rightarrow T)(v_2, v_3) = 0$$
$$R(u_1, v_3) \circ (R \Rightarrow T)(v_3, v_3) = 0.9 + 0.6 - 1 = 0.5$$
Hence, \( Z_{3,3} = \{ \begin{pmatrix} 0.8 \\ 0 \\ 0 \end{pmatrix} \} \).

For \((R \circ (R \Rightarrow T))(u_2, w_3) = 0.3, \) we have

\[
R(u_2, v_1) \circ (R \Rightarrow T)(v_1, w_3) = 0.2 + 0.8 - 1 = 0
\]

\[
R(u_2, v_2) \circ (R \Rightarrow T)(v_2, w_3) = 0.9 + 0.4 - 1 = 0.3
\]

\[
R(u_2, v_3) \circ (R \Rightarrow T)(v_3, w_3) = 0.7 + 0.6 - 1 = 0.3
\]

two possibilities providing two columns: \( Z_{2,3} = \{ \begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix} \} . \)

For \((R \circ (R \Rightarrow T))(u_3, w_3) = 0.6 \) we have

\[
R(u_3, v_1) \circ (R \Rightarrow T)(v_1, w_3) = 0.8 + 0.8 - 1 = 0.6
\]

\[
R(u_3, v_2) \circ (R \Rightarrow T)(v_2, w_3) = 0.6 + 0.4 - 1 = 0
\]

\[
R(u_3, v_3) \circ (R \Rightarrow T)(v_3, w_3) = 0.9 + 0.6 - 1 = 0.5
\]

Therefore, \( Z_{3,3} = \{ \begin{pmatrix} 0.8 \\ 0 \\ 0 \end{pmatrix} \} \).

In this case, there are two minimal covering of the set \( Z_3 = \{ Z_{1,3}, Z_{2,3}, Z_{3,3} \} : \)

\[
(0.8) = (0.8) \lor (0.4) \lor (0.6)
\]

This yields four fuzzy relations, defined as follows

\[
X_4 \quad w_1 \quad w_2 \quad w_3
\]

\[
\begin{array}{ccc}
 v_1 & 0 & 0 & 0.8 \\
v_2 & 0 & 0.8 & 0.4 \\
v_3 & 0.9 & 0.5 & 0 \\
\end{array}
\]

\[
X_5 \quad w_1 \quad w_2 \quad w_3
\]

\[
\begin{array}{ccc}
 v_1 & 0 & 0 & 0.8 \\
v_2 & 0 & 0.8 & 0 \\
v_3 & 0.9 & 0.5 & 0.6 \\
\end{array}
\]

\[
X_6 \quad w_1 \quad w_2 \quad w_3
\]

\[
\begin{array}{ccc}
 v_1 & 0 & 0.5 & 0.8 \\
v_2 & 0 & 0.8 & 0.4 \\
v_3 & 0.9 & 0 & 0 \\
\end{array}
\]

\[
X_7 \quad w_1 \quad w_2 \quad w_3
\]

\[
\begin{array}{ccc}
 v_1 & 0 & 0.5 & 0.5 \\
v_2 & 0 & 0.8 & 0 \\
v_3 & 0.9 & 0.6 & 0 \\
\end{array}
\]

that solve Equation (2). By their construction and the properties of the Łukasiewicz conjunctor, they are minimal solutions.

**Example III.2.** In this example, we consider the Gödel structure [9], then \( L = [0, 1] \) and \( \odot \colon L \times L \rightarrow L \) and \( \rightarrow \colon L \times L \rightarrow L \) are defined by \( x \odot y = \min \{ x, y \} \) and \( z \rightarrow w = \begin{cases} 1 & \text{if } y \leq z \\ z & \text{otherwise} \end{cases} \)

for all \( x, y, z \in [0, 1] \). Given \( U = \{ u_1, u_2 \}, V = \{ v_1, v_2, v_3 \}, W = \{ w \} \) and

\[
R \quad v_1 \quad v_2 \quad v_3 \\
\begin{array}{ccc}
 u_1 & 0.6 & 0.4 & 0.5 \\
u_2 & 0.8 & 0.7 & 0.6 \\
u_3 & 0.9 & 1 & 0.9 \\
\end{array}
\]

\[
T \quad w \\
\begin{array}{c}
 u_1 & 0.6 \\
u_2 & 0.7 \\
u_3 & 0.9 \\
\end{array}
\]

the direct computation shows that

\[
R \Rightarrow T \quad w \\
\begin{array}{c}
 v_1 & 0.7 \\
v_2 & 0.9 \\
v_3 & 1.0 \\
\end{array}
\]

is the maximal solution of Equation (2). In order to verify the equality \( (R \circ (R \Rightarrow T))(u_1, w) = 0.6 \), we compute

\[
R(u_1, v_1) \circ (R \Rightarrow T)(v_1, w) = 0.6 \land 0.7 = 0.6
\]

\[
R(u_1, v_2) \circ (R \Rightarrow T)(v_2, w) = 0.4 \land 0.9 = 0.4
\]

\[
R(u_1, v_3) \circ (R \Rightarrow T)(v_3, w) = 0.5 \land 1.0 = 0.5
\]

Note that we only need the value associated with \( v_1 \). Moreover, this value can be reduced until 0.6. Hence, the first (and only) column of a solution has to be contained in the following set

\[
Z_{1,1} = \{ \begin{pmatrix} x \\ 0 \end{pmatrix} \mid 0.6 \leq x \leq 1 \}
\]

Focusing on our main goal, the least one is the column associated with a minimal solution. Hence, we only consider the column \( Z_{1,1} = \{ \begin{pmatrix} 0.6 \\ 0 \end{pmatrix} \} \).

For \((R \circ (R \Rightarrow T))(u_2, w) = 0.7 \) we have

\[
R(u_2, v_1) \circ (R \Rightarrow T)(v_1, w) = 0.8 \land 0.7 = 0.7
\]

\[
R(u_2, v_2) \circ (R \Rightarrow T)(v_2, w) = 0.7 \land 0.9 = 0.7
\]

\[
R(u_2, v_3) \circ (R \Rightarrow T)(v_3, w) = 0.6 \land 1.0 = 0.6
\]

Now the values associated with \( v_1 \) and \( v_2 \) provide the maximum. Furthermore, the value for \( v_2 \) can also be decreased, specifically, any element \( x \) in \([0.7, 1]\) provides the same maximum result: \( 0.7 \land x = 0.7 \). Therefore, focusing on the minimal solutions we only need to consider:

\[
Z_{2,1} = \{ \begin{pmatrix} 0.7 \\ 0 \end{pmatrix} \}
\]

Finally, for \((R \circ (R \Rightarrow T))(u_3, w) = 0.9 \) we calculate

\[
R(u_3, v_1) \circ (R \Rightarrow T)(v_1, w) = 0.9 \land 0.7 = 0.7
\]

\[
R(u_3, v_2) \circ (R \Rightarrow T)(v_2, w) = 1.0 \land 0.9 = 0.9
\]

\[
R(u_3, v_3) \circ (R \Rightarrow T)(v_3, w) = 0.9 \land 1.0 = 0.9
\]

In this last case \( v_2 \) and \( v_3 \) are involved in the computation of the maximum and the value associated with \( v_3 \) can be decreased until 0.9. These considerations yield the following minimal solutions

\[
\begin{pmatrix} 0.6 \\ 0.7 \\ 0.9 \end{pmatrix}
\]

\[
\begin{pmatrix} 0.7 \\ 0.9 \end{pmatrix}
\]

\[
\begin{pmatrix} 0.6 \\ 0.9 \end{pmatrix}
\]

**IV. Conclusion and future works**

The main aim of this research is to define as generally as possible an algebraic structure that allows the existence of minimal solutions of the fuzzy relation equations defined based on this structure. For that, a general increasing operation \( \odot \), which only satisfies the adjointness property, i.e. is residuated, and satisfies the IPNE-condition, has been considered to define a general fuzzy relation equation, which has minimal solutions whenever a solution exists. Moreover, a new algebraic characterization using the notion of covering is introduced,
which provides a method to obtain the minimal solutions and, consequently, the whole set of solutions. As future work, the obtained results will be applied to several problems in fuzzy logic, such as to abduction reasoning. It is well-known that implications in MV-algebras are infinitely distributive. A topic of future study is to characterize all structures where implication is infinitely distributive. Algebraic structures that satisfy the INPE-condition are not studied much; also they will be a topic of future research.

REFERENCES


