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A Stochastic Optimization Method for European Option Pricing

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***Abstract*—In the contemporary finance the Monte Carlo and quasi-Monte Carlo methods are solid instruments to solve various problems. In the paper the problem of deriving the fair value of European style options is considered. Regarding the option pricing problems, Monte Carlo methods are extremely efficient and useful, especially in higher dimensions. In this paper we show simulation optimization methods which essentially improve the accuracy of the standard approaches for European style options.**

I. INTRODUCTION

THE Monte Carlo approach has become a popular computational tool to solve problems in quantitative finance [3]. New approaches have been designed to outperform classical Monte Carlo ones in terms of numerical efficiency. It is observed that there could be efficiency gains in using special optimization stochastic approaches instead of the random sequences distinctive for standard Monte Carlo [10]. The basic definitions are taken form [5], [6], [8], [23], [25].

A European call option provides its holder with the right, but not the obligation, to byu some quantity of a prescribed asset (underlying) S at a prescribed price (strike or exercise price) E at a prescribed time (maturity or expiry date) T.

A European put option has the same features as its call counterpart, except that holder could sell the underlying rather than buying it.

Risk neutrality is a feature of an investor who is indifferent to the quantity of risk. This definition takes part in the formation of the risk-neutral evaluation formula. A more rigourous approach, based on an appropriately defined probability space of random variables could be found in [9], [17].

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The risk-free interest rate r is an abstract interest rate, used to borrow or lend money at, which is sometimes implied from the yields of T-bonds.

The Wiener process $\mathrm{d}X$ is a special type of Markov stochastic process with the respective properties: $\mathrm{d}X \sim \mathrm{N}(0, \sqrt{\mathrm{d}t})$, where $\mathrm{N}(\mu, \sigma)$ is the Gaussian distribution with mean μ and variance σ^2.

The most fundamental problem in option pricing is obtaining a "fair" value of the option contract $V(S,t)$, if the following parameters are given: the exercise price E, the asset price $S(t)$, the expiry time T, the risk-free interest rate r and the assumption on the dynamics model of S:

$$\mathrm{d}S = \mu S \mathrm{d}t + \sigma S \mathrm{d}X, \tag{1}$$

where $\mathrm{d}X$ is the increment of a Wiener process, μ is the drift rate and σ is the volatility of the asset price, measuring the average growth and level of fluctuations, respectively.

The celebrated Black-Scholes pricing formula for a European call option can be written ([2] or [25]) using the following parabolic partial differential equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{2}$$

with final condition

$$V(S,T) = \max(S - E, 0)$$

and Dirichlet boundary conditions

$$V(0,t) = 0, V(S,t) \sim S - Ee^{-r(T-t)}, S \to \infty.$$

The European put option price is governed by the same equation as (2), but with terminal condition

$$V(S,T) = \max(E - S, 0),$$

and boundary conditions

$$V(0,t) = Ee^{-r(T-t)}, V(S,t) \sim 0, S \to \infty.$$

Fortunately, there exist explicit closed form solutions. The call option is described by

$$V(S,t) := C(S,t) = S\mathrm{N}(d_1) - Ee^{-r(T-t)}\mathrm{N}(d_2),$$

where

$$d_1 = \frac{\ln(\frac{S}{E}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_2 = \frac{\ln(\frac{S}{E}) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

and $\mathrm{N}(z)$ is the cumulative distribution function of the standard normal distribution. Considering put option,

$$V(S,t) := P(S,t) = Ee^{-r(T-t)}\mathrm{N}(-d_2) - S\mathrm{N}(-d_1),$$

where d_1, d_2, and $\mathrm{N}(z)$ remain the same.

There are two general approaches in Monte Carlo modelling [7]. The first one is *Monte Carlo simulation*, where the algorithms are used for direct simulation of the underlying phenomena of the model. Thus Monte Carlo serves as a tool to choose from the many possible outcomes in a particular circumstance, and it solves probabilistic problems simulating random variables and processes. The other direction is *Monte Carlo numerical* methods, where the algorithms are used to solve deterministic methods by manufacturing random variables. In the context of option pricing, the basic idea is to represent the option premium as mathematical expectation of the random variable (European option risk-neutral valuation formula [17]):

$$V(S,t) = \mathbb{E}_{\mathbb{Q}}(e^{-r(T-t)}h(S(T)) \mid S(t) = S, \mu = r)$$

where $\mathbb{E}_{\mathbb{Q}}(\cdot)$ is the expectation operator, $h(S)$ is the payoff function, in particular $h(S) = \max(S - E, 0)$ for a call and $h(S) = \max(E - S, 0)$ for a put.

In this paper, we employ the first approach of Monte Carlo simulation (of the Geometric Brownian motion (1)).

II. Stochastic methods

One of the basic problems of the methods belonging to Monte Carlo family is the fact that one realization which is close to other ones does not bring a lot of new information for the underlying problem. This could be solved by applying variance-reduction techniques, one of which is the stratified sampling. It splits the original integration domain in several subdomains. It could be shown that the variance of stratified sampling is never greater that the crude method sampling [24].

Latin Hypercube Sampling (LHS) is one of most pronounced types of stratified sampling, originating from [20]. Similar procedure is described in [11]. An improved version of LHS is proposed by [21], [22]. We briefly describe the method in terms of numerical integration. The whole domain $[0,1]^d$ is split into m^d disjoint subdomains of volume m^{-d}, and then a single point is sampled from each subdomain. Let the point x_k, $k = 1, \ldots, m^d$ has coordinates $\mathrm{x}_{k,j}$, $j = 1, \ldots, d$. One of the advantages of LHS is that it does not require more samples in case of more dimensions. A sample scheme of random, stratified and LHS in case of sixteen points is given on Fig. 1 [15].

We supply a weaker variant of the theorem from [20]:

Theorem 1: If $f(\mathrm{x}_1, \mathrm{x}_2, \ldots \mathrm{x}_d)$ is a monotonic function w. r. t. each of its arguments, then $\mathrm{Var}(T_L) \leq \mathrm{Var}(T_R)$, where T_L is the approximation of $\int_{U^d} f(\mathbf{x})\mathrm{d}\mathbf{x}$ derived by the LHS method and T_R is the approximation of the latter integral derived by random sampling.

Let $\mathrm{x}_i = (x_i^{(1)}, x_i^{(2)}, \ldots, x_i^{(s)}),\ i = 1, 2, \ldots$. Then the discrepancy (star discrepancy) of the set is provided by the formula:

$$D_N^* = D_N^*(\mathrm{x}_1, \ldots, \mathrm{x}_N) = \sup_{\Omega \subset \mathrm{E}^s} \left| \frac{\#\{\mathrm{x}_n \in \Omega\}}{N} - V(\Omega) \right|,$$

where $\mathrm{E}^s = [0,1)^s$.

Let the base b representation of n be given by the expression [18]: $n = \ldots a_3(n), a_2(n), a_1(n),\ n > 0, n \in \mathbb{Z}$.

Let the radical inverse sequence be defined as [19]:

$n = \sum_{i=0}^{\infty} a_{i+1}(n)b^i,\ \ \phi_b(n) = \sum_{i=0}^{\infty} a_{i+1}(n)b^{-(i+1)}$ and its discrepancy satisfy: $D_N^* = \mathcal{O}\left(\frac{logN}{N}\right)$. If $b = 2$, then the Van der Corput sequence [19] is derived.

Halton sequence [13], [14] is defined as:

$$s_n^{(k)} = \sum_{i=0}^{\infty} \sigma_{i+1}^{(k)} a_{i+1}^{(k)}(n) b_k^{-(i+1)},$$

where $(b_1, b_2, \ldots, b_s) \equiv (2, 3, 5, \ldots, p_s)$, where p_i designates the i-th prime, and $\sigma_i^{(k)}$, $i \geq 1$ is the set of permutations on $(0, 1, 2, \ldots, p_k - 1)$.

Sobol sequence [1], [4], [12] is defined as:

$$\mathrm{x}_k \in \overline{\sigma}_i{}^{(k)}, k = 0, 1, 2, \ldots,$$

where $\overline{\sigma}_i{}^{(k)}$, $i \geq 1$ is the set of permutations on each $2^k (k = 0, 1, 2, \ldots)$ subsequent points of the Van der Corput sequence. In case of binary we arrive on:

$x_n^{(k)} = \bigoplus_{i \geq 0} a_{i+1}(n)v_i$, where v_i, $i = 1, \ldots, s$ is a set of direction numbers [16].

The following discrepancy estimate is valid for the ***Halton, Sobol, Faure*** -based QMC algorithms:

$$D_N^* = \mathcal{O}\left(\frac{log^s N}{N}\right).$$

The used implementation of the Sobol sequence (SOB) in this paper is an adaptation of the idea of [1] and further modifies the INSOBL and GOSOBL procedures in ACM TOMS Algorithm 647 [12] and ACM TOMS Algorithm 659 [4], [16].

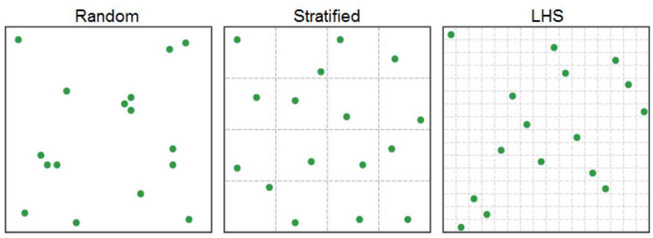

Fig. 1. Example of random, stratified and LHS with sixteen points (d = 2, m = 4).

The standard M-dimensional Halton sequence (HAL) [13], [14] comprises M 1-dimensional van der Corput sequences and uses first M primes as bases.

Another variance reduction techniques consist of manipulating the (quasi-)random sequence in such a way to improve its desired probabilistic properties.

In a simple Monte Carlo simulation, the samples are independent and identically distributed. The idea of **antithetic variates** is to reduce the variance by introducing samples that are negatively correlated.

The concept of **moment matching** Monte Carlo follows the idea that the generated samples from the Monte Carlo method are expected to obey the same statistical properties as the theoretical distribution. This simply means that the empirical moments of the sample should equal their theoretical counterparts. It may seem appealing at first glance, but it raises concerns as well, the primary of which is that the distribution is no longer the same.

III. Numerical examples and results

In this section we will present some computational experiments and simulations in order to demonstrate the effectiveness of the proposed approach. We consider a European call option with $r = 0.05$, $\sigma = 0.2$, $K = 220$ and $T = 1$ year time to maturity. We assume that $S_0 = 200$.

For reference values, we provide the results with the crude / plain Monte Carlo method (Table I).

TABLE I
Crude Monte Carlo method

N	V_{exact}	V	Error	Time (sec)
1e2	12.0801762594485	11.395	0.68508	0.009731
1e3	12.0801762594485	11.54	0.54011	0.001870
1e4	12.0801762594485	12.045	0.034958	0.002909
1e5	12.0801762594485	12.159	0.078974	0.006350
1e6	12.0801762594485	12.114	0.033989	0.043681
1e7	12.0801762594485	12.089	0.0092241	0.326507
1e8	12.0801762594485	12.082	0.0016232	3.922560

A. Latin hypercube sampling

First we give the results with the LHS quasirandom sequence. For different number of random points, distributed logequally from $N = 1e2$ to $N = 1e8$, we display the results from the application of the sole LHS and consecutively combined with antithetic variates and moment matching techniques, see Tables II, III and IV.

TABLE II
LHS without other variance reduction

N	V_{exact}	V	Error	Time (sec)
1e2	12.0801762594485	12.01	0.070212	0.001102
1e3	12.0801762594485	12.064	0.01657	0.001364
1e4	12.0801762594485	12.083	0.0026497	0.002349
1e5	12.0801762594485	12.08	5.5395e-05	0.010674
1e6	12.0801762594485	12.08	6.7257e-06	0.097699
1e7	12.0801762594485	12.08	2.7833e-08	0.992030
1e8	12.0801762594485	12.08	1.6933e-07	12.402776

TABLE III
LHS with antithetic variates

N	V_{exact}	V	Error	Time (sec)
1e2	12.0801762594485	12.029	0.051276	0.001545
1e3	12.0801762594485	12.077	0.0031408	0.001317
1e4	12.0801762594485	12.08	0.00021551	0.002592
1e5	12.0801762594485	12.08	4.762e-05	0.011522
1e6	12.0801762594485	12.08	1.4455e-05	0.107597
1e7	12.0801762594485	12.08	1.975e-07	1.083230
1e8	12.0801762594485	12.08	9.3442e-08	14.027701

TABLE IV
LHS with moment matching

N	V_{exact}	V	Error	Time (sec)
1e2	12.0801762594485	12.027	0.052906	0.002450
1e3	12.0801762594485	12.067	0.01366	0.001866
1e4	12.0801762594485	12.08	0.00027461	0.003071
1e5	12.0801762594485	12.08	6.2304e-05	0.010709
1e6	12.0801762594485	12.08	6.9669e-06	0.102625
1e7	12.0801762594485	12.08	1.2712e-06	1.034859
1e8	12.0801762594485	12.08	1.2036e-07	14.027701

In general, the application of variance reduction techniques takes 10%-15% more computational time. For small values of N, the combined approach gives more accurate results that the plain Halton sequence. For bigger values of N, this advantage is not that pronounced, as the antithetic variates technique appears to perform better than the moment matching one.

B. Halton sequence

We continue the simulations with the Halton sequence. The results concerning the sole Halton and the additionally applied antithetic variates and moment matching techniques are presented on Tables V, VI and VII.

TABLE V
Halton without other variance reduction

N	V_{exact}	V	Error	Time (sec)
1e2	12.0801762594485	11.111	0.96902	0.004395
1e3	12.0801762594485	12.099	0.01904	0.005186
1e4	12.0801762594485	12.073	0.0068783	0.010478
1e5	12.0801762594485	12.081	0.00043392	0.064871
1e6	12.0801762594485	12.08	0.00010187	0.628980
1e7	12.0801762594485	12.08	2.9878e-05	6.310790
1e8	12.0801762594485	12.08	5.5002e-06	65.628562

TABLE VI
Halton with antithetic variates

N	V_{exact}	V	Error	Time (sec)
1e2	12.0801762594485	12.298	0.21807	0.002178
1e3	12.0801762594485	12.056	0.024416	0.003771
1e4	12.0801762594485	12.082	0.0021855	0.009744
1e5	12.0801762594485	12.08	0.00034855	0.065575
1e6	12.0801762594485	12.08	3.5754e-05	0.627989
1e7	12.0801762594485	12.08	2.9969e-05	6.493375
1e8	12.0801762594485	12.08	7.532e-06	67.408394

The conclusions about the Halton sequence are similar to these of the LHS. For lower values of N, the results are relatively bad. For higher, however, the accuracy improves

TABLE VII
HALTON WITH MOMENT MATCHING

N	$V_{\rm exact}$	V	Error	Time (sec)
1e2	12.0801762594485	11.669	0.41132	0.002109
1e3	12.0801762594485	12.084	0.0037734	0.003770
1e4	12.0801762594485	12.081	0.00079293	0.010174
1e5	12.0801762594485	12.08	3.4485e-05	0.064448
1e6	12.0801762594485	12.08	1.5061e-05	0.636945
1e7	12.0801762594485	12.08	4.5434e-07	6.388943
1e8	12.0801762594485	12.08	1.3984e-06	66.411237

significantly and the moment matching technique performs best.

C. Sobol sequence

We conclude the experiments with tests with the Sobol quasirandom sequence. The simulations follow the previous pattern and the results are given in Tables VIII, IX and X.

TABLE VIII
SOBOL WITHOUT OTHER VARIANCE REDUCTION

N	$V_{\rm exact}$	V	Error	Time (sec)
1e2	12.0801762594485	11.171	0.90959	0.003620
1e3	12.0801762594485	12.062	0.018076	0.006366
1e4	12.0801762594485	12.095	0.015131	0.008283
1e5	12.0801762594485	12.083	0.002552	0.051577
1e6	12.0801762594485	12.082	0.0015908	0.447782
1e7	12.0801762594485	12.08	0.00017189	4.461646
1e8	12.0801762594485	12.08	6.1323e-05	44.393839

TABLE IX
SOBOL WITH ANTITHETIC VARIATES

N	$V_{\rm exact}$	V	Error	Time (sec)
1e2	12.0801762594485	11.021	1.0594	0.002160
1e3	12.0801762594485	12.125	0.044527	0.002771
1e4	12.0801762594485	12.091	0.010636	0.008956
1e5	12.0801762594485	12.083	0.0030839	0.050479
1e6	12.0801762594485	12.082	0.0016988	0.451211
1e7	12.0801762594485	12.08	0.00015271	4.522295
1e8	12.0801762594485	12.08	5.9658e-05	45.655215

TABLE X
SOBOL WITH MOMENT MATCHING

N	$V_{\rm exact}$	V	Error	Time (sec)
1e2	12.0801762594485	12.05	0.030024	0.002263
1e3	12.0801762594485	12.001	0.079036	0.003115
1e4	12.0801762594485	12.093	0.012791	0.007114
1e5	12.0801762594485	12.08	0.0001421	0.049672
1e6	12.0801762594485	12.08	0.00012572	0.467593
1e7	12.0801762594485	12.08	5.1817e-05	4.473896
1e8	12.0801762594485	12.08	1.0556e-05	45.425531

Again the general implications remain valid. The antithetic variates performance does not differ significantly from the plain Sobol one, while they are outperformed by the moment matching technique.

Finally, the best results are obtained by the combination of Latin hypercube sampling with the antithetic variates.

IV. CONCLUSION

An efficient optimization technique for Monte Carlo methods has been presented. The proposed approach shows an essential advantage over the well known stochastic approaches based on the standard sequences. The improved accuracy will be crucial for more reliable results for European option pricing. What is more, the suggested approach could be used in situations where the other deterministic methods fail – e. g. in case of high dimensions, complex contract specifications, etc.

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