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# On Multiplicative, Additive and Qualitative Pairwise Comparisons

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***Abstract*—A relationship between the classical multiplicative pairwise comparisons that are based on $a_{ij}a_{ji} = 1$, the additive model based on $b^i_j + b^j_i = 1$, and qualitative pairwise comparisons that uses the relations $\approx$, $\sqsubset$, $\subset$, $<$ and $\prec$, is discussed in detail. A special attention is paid to the concept of consistency and weights calculations. An on-line tool is also discussed.**

***Index Terms*—pairwise comparisons, consistency, weights calculation, qualitative judgements**

## I. Introduction

The *pairwise comparisons* method is based on the observation that it is much easier to rank the importance of *two* objects than it is to rank the importance of *several* objects. This very old idea goes back to Ramon Llull in the end of XIII century. Its modern version is due to 1785 influential paper by Marquis de Condorcet and was later developed by Fechner (1860) and Thurstone (1927) [9], [18]. The modern version is usually associated with Saaty's AHP (Analytical Hierarchy Process) [17].

Classical pairwise comparisons can be called *multiplicative* [6] as a coefficient $a_{ij}$ is interpreted as an entity $E_i$ *is* $a_{ij}$ *times preferred than an entity* $E_j$. Alternatively we may define *additive* pairwise comparisons where a coefficient $b^i_j$ is interpreted as $b^i_j$ *measures the importance of* $E_i$ *in comparison with* $E_j$ *assuming that their total importance is* 1.0 (*or* 100%) [3], [6].

When mostly subjective judgment is involved, providing immediately reasonable quantitative relationship between two entities is usually difficult if not almost impossible. We usually start with some qualitative (relational) judgment like '$E_i$ *is only slightly better than* $E_j$', etc., so we need some good qualitative (relational) model as well [5], [9], [8]. In many cases the use of *combined* pairwise comparisons, that involve simultaneous use of multiplicative, additive and qualitative versions is the best and recommended solution [6], [7], [16].

In this paper we will provide a detailed comparison of multiplicative and additive pairwise comparisons, including the concept of consistency and optimal weights assignment. Since in reality, practically every pairwise comparisons process starts with some qualitative estimations, the qualitative pairwise comparisons [5], [9], [8] and their relationship to multiplicative and additive models is also discussed in detail. We will also provide an easy to use on-line tool, called PiXR, that makes use of the method presented in the paper, rather easy.

## II. Multiplicative and Additive Pairwise Comparisons

Let $E_1, ..., E_n$ be a finite set of objects (entities) to be judged and/or analyzed. The quantitative relationship between entities $E_i$ and $E_j$ is represented by a positive number $a_{ij}$. We assume $a_{ij} > 0$ and $a_{ij} = \frac{1}{a_{ji}}$, for $i, j = 1, ..., n$ (which implies $a_{ii} = 1$ for all $i$). If $a_{ij} > 1$ then $E_i$ is more important (preferred, better, etc.) than $E_j$ and $a_{ij}$ is a measure of this relationship (the bigger $a_{ij}$, the bigger the difference), if $a_{ij} = 1$ then $E_i$ and $E_j$ are indifferent. We call this model *multiplicative* since $a_{ij}$ is interpreted as $E_j$ *is* $a_{ij}$ *times preferred* (*more important, etc.*) *than* $E_j$.

The matrix of such (multiplicative) relative comparison coefficients, $A = [a_{ij}]_{n\times n}$, is called a (multiplicative) *pairwise comparison matrix* [17].

A pairwise comparison matrix $A = [a_{ij}]_{n\times n}$ is *consistent* [17] if and only if

$$a_{ij}a_{jk} = a_{ik}, \tag{1}$$

for $i, j, k = 1, ..., n$. Saaty's Theorem [17] states that a pairwise comparison matrix $A$ is consistent if and only if there exist positive numbers $w_1, ..., w_n$ such that $a_{ij} = w_i/w_j$, $i, j = 1, ..., n$. The values $w_i$ are unique up to a multiplicative constant. They are often called *weights* and interpreted as a measure of importance. Weights may be scaled to $w_1 + .. + w_n = 1$ (or 100%) and they obviously create 'natural' ranking ($E_i < E_j \iff w_i < w_j$ and $E_i \approx E_j \iff w_i = w_j$). In practice, the values $a_{ij}$ are very seldom consistent so some measurements of inconsistency are needed. Saaty [17] proposed an inconsistency index based on the value of the largest eigenvalue of $A$. The basic problem is that *it does not give any clue where most inconsistent values of* $A$ *are located* [2], [9], [10]. On the other hand, *distance-based inconsistency* [10], for a given $A = [a_{ij}]_{n\times n}$, defined as:

$$cm_A = \max_{(i,j,k)} \left( \min \left( \left| 1 - \frac{a_{ij}}{a_{ik}a_{kj}} \right|, \left| 1 - \frac{a_{ik}a_{kj}}{a_{ij}} \right| \right) \right) \tag{2}$$

localizes the most inconsistent triad, so we can reduce inconsistency by some minor changes $a_{ij}, a_{ik}, a_{kj}$. Recently a fast algorithm for inconsistency reduction has been proposed [11].

To find suitable values of weights from an inconsistent, but with acceptable level of inconsistency, matrix $A$, one can either calculate the principal eigenvector of the matrix $A$ [17], or use the geometric means of columns (or equivalently, rows) of the matrix $A$ [1] (i.e. for $i = 1, ..., n$, $w_i = \sqrt[n]{\prod_{j=1}^{n} a_{ij}}$). For small values of the inconsistency index, both methods produce very similar results [2].

When applying pairwise comparisons to various problems we have noticed that, especially when the entities $E_i$ and $E_j$ were not much different, experts felt often much more comfortable and more confident when they were asked to divide 100 quality points between entities $E_i$ and $E_j$ than to provide multiplicative relationship [6], [7], [16], i.e. ratio $a_{ij}$. Dividing of 100 between $E_i$ and $E_j$ means that we are replacing the multiplicative relationship $a_{ij}a_{ji} = 1$, with the *additive* relationship $b^i_j + b^j_i = 1$.

In this approach, we model the mutual relationship between $E_i$ and $E_j$ by two numbers $b^i_j$ and $b^j_i$, where: $b^i_j$ measures the importance of $E_i$ in comparison with $E_j$ assuming that their total importance is 1.0 (or 100%), and similarly for $b^j_i$. Formally we assume that, for all $i, j = 1, \ldots, n$,

$$b^i_j \geq 0, b^j_i \geq 0 \text{ and } b^i_j + b^j_i = 1 \tag{3}$$

Clearly $b^i_i = 0.5$ (or 50%) for all $i = 1, \ldots, n$.

The matrix of such additive relative comparison coefficients, $B = [b^i_j]_{n\times n}$, is called an *additive pairwise comparison matrix*.

When $b^i_j$ is interpreted as the probability that judges would prefer the entity $E_i$ over $E_j$, the equation (3) is exactly the same as in Bradley-Terry model [3].

To transform additive model into standard multiplicative one, we need a mapping $\phi : \langle 0, 1\rangle \rightarrow \langle 0, \infty)$ such that $a+b = 1$ implies $\phi(a)\cdot\phi(b) = 1$. When $a$ and $b$ are interpreted as *two parts of one whole* (which equals 1.0, or 100%), then $\frac{a}{b}$ represents *ratio* between $a$ and $b$, and $\frac{a}{1-a}$ represents ratio between $a$ and its complement.

Hence, the most natural mapping seems to be $\phi(a) = \frac{a}{1-a}$. This mapping has many different applications [13], and in our case leads to the transformation of 'additive' model into 'multiplicative' model [6], [9], [8].

For all $i, j = 1, \ldots, n$:

$$a_{ij} = \frac{b^i_j}{1 - b^i_j} = \frac{b^i_j}{b^j_i} \tag{4}$$

From equation (4) we immediately get that for all $i, j = 1, \ldots, n$, we have:

$$b^i_j = \frac{a_{ij}}{a_{ij} + 1} \quad \text{and} \quad a_{ij}a_{ji} = 1 \iff b^i_j + b^j_i = 1 \tag{5}$$

We may now analyze and reduce inconsistency by using the formulas for multiplicative case.

## III. Qualitative and Combined Pairwise Comparisons

Instead of numerical values $a_{ij}$ or $b^i_j$, the binary relations $\approx$, $\sqsubset$, $\subset$, $<$, $\prec$ and $\sqsupset$, $\supset$, $>$, $\succ$ over the set of entities $Ent = \{E_1, \ldots, E_n\}$ are used [5], [8]. The relations are interpreted as

- $a \approx b$ : $a$ and $b$ are *indifferent*,
- $a \sqsubset b$ : *slightly in favor* of $b$,
- $a \subset b$ : *in favour* of $b$,
- $a < b$: $b$ is *strongly better*,
- $b$ is *extremely better*.

The tuple $(Ent, \approx, \sqsubset, \subset, <, \prec)$ is called *qualitative pairwise comparisons systems*. The number of relations has been limited to five because of the known restrictions of human mind when it comes to subjective judgments [4], [15]. The above relations are disjoint and cover all the cases and the relation $\approx$ is symmetric and includes identity.

In this case the consistency is defined by a set of 45 axioms [8] that consider all relational compositions of the above relations. The idea behind all these axioms is very simple and natural:

*composition of relations should be relatively continuous and must not change preferences in a drastic way.*

Consider the following composition of preferences: $a \approx b \wedge b \sqsubset c$. What relationship between $a$ and $c$ is consistent? Intuitively, $a \approx c$ and $a \sqsubset c$ are for sure consistent, $a \subset c$ is debatable, while $a < c$ and $a \prec c$ are definitively inconsistent. This reasoning leads to the Axiom 2.1 [9]:

2.1 $(a \approx b \wedge b \sqsubset c) \vee (a \sqsubset b \wedge b \approx c) \implies (a \approx c \vee a \sqsubset c \vee a \subset c)$. There are two algorithms that remove inconsistency for qualitative pairwise comparisons [8].

Define the relations $\widehat{\prec} = \prec$, $\widehat{<} = \prec \cup <$, $\widehat{\subset} = \prec \cup < \cup \subset$ and $\widehat{\sqsubset} = \prec \cup < \cup \subset \cup \sqsubset$. If the qualitative pairwise comparison system is consistent [9], [8], then the relations $\widehat{\prec}$, $\widehat{<}$, $\widehat{\subset}$ and $\widehat{\sqsubset}$ are (sharp) partial orders.

In reality, almost always we start with some qualitative relationship, then we try to assign some reasonable number ($a_{ij}$ or $b^i_j$) to it, and after this we provide some consistency analysis [6], [7], [16]. In some cases we go back to qualitative relationship and eventually analyze the case again from the beginning [6], [7], [16]. Finding proper numbers corresponding to qualitative relations is usually tricky as a trusted methodology still does not exists. Usually numbers $1, 2, \ldots, 5$ [10] or $1, 2, \ldots, 10$ [17] are proposed as initial attempts, without much justification except limits of human mind [4], [15]. More systematic approach involves the concept of qualitative consistency [9], [8]. It was proven that if the $b^i_j$'s or $a_{ij}$'s are assigned to relations $R_{ij}$ as shown in Table I, then the qualitative pairwise comparisons are consistent [9]. In fact, having a result in opposite direction would be more desirable but this is still an open problem. Nevertheless using Table I proved useful as it usually results in decent inconsistency [6], [7], [16].

The columns 1 and 3 contain intervals, to start the process one needs to pick one point from appropriate interval, and in general a choice is not obvious. To help users without experience default values are proposed in columns 2, 4 and 5. The column 2 contains just mean values of each interval, the column 4 contains the values of $a_{ij}$s derived from appropriate values of column 2 interpreted as $b^i_j$s, and the column 5

TABLE I
RELATIONSHIP BETWEEN *arithmetic*, *geometric* AND *relational* SCALES AS PROPOSED BY JANICKI AND ZHAI [9]. WE HAVE HERE $a_{ij} = \frac{b^i_j}{1-b^i_j}$.

| *arithmetic scale* | | *geometric scale* | | | *rel. scale* | Definition of importance ($E_i$ vs $E_j$) |
|---|---|---|---|---|---|---|
| range of $b^i_j$ | | range of $a_{ij}$ | | | relation $R_{ij}$ | |
| range | default value | range | default value 1 | default value 2 | | |
| 0.44-0.55 | 0.5 | 0.79-1.27 | 1.0 | 1.0 | $\approx$ | *indifferent* |
| 0.56-0.65 | 0.6 | 1.28-1.94 | 1.5 | 1.6 | $\sqsupset$ | *slightly in favour* |
| 0.66-0.75 | 0.7 | 1.95-3.17 | 2.3 | 2.6 | $\sqsupset$ | *in favour* |
| 0.76-0.85 | 0.8 | 3.18-6.14 | 4.0 | 4.7 | $>$ | *strongly better* |
| 0.86-1.00 | 0.9 | 6.15- | 9.0 | 7.0 | $\succ$ | *extremely better* |

contains mean values of the intervals from column 3. When transforming initial relations $R_{ij}$, what value should we attach to appropriate $a_{ij}$, the one from column 4 (default 1) or from column 5 (default 2)? This problem will be discussed in detail the next section, however the difference is not a big one. When a choice is left to the users, usually default value 2 is used [6], [16].

## IV. CONSISTENCY FOR ADDITIVE PAIRWISE COMPARISONS

In this section we will analyze an intuitively natural concept of consistency for additive pairwise comparisons.

Let $B = [b^i_j]_{n\times n}$ be an additive pairwise comparison matrix. Consider $b^i_k$, $b^k_j$ and $b^i_j$. We have $b^i_k + b^k_i = b^k_j + b^j_k = b^i_j + b^j_i = 1$. How can we decide if $b^i_k$, $b^k_j$ and $b^i_j$ are consistent or not? The values of $b^i_k$ and $b^k_j$ alone do not seem to have any reasonable relations to the value of $b^i_j$. The value of $b^i_k$ just indicates the part of one that is assigned to the entity $E_i$ when $E_i$ is compared with $E_k$, so it can hardly be compared to or associated with $b^i_j$. However we may try to use ratios $\frac{b^i_k}{b^k_i}$, $\frac{b^k_j}{b^j_k}$ and $\frac{b^i_j}{b^j_i}$ to define a relationship between the entities $E_i$, $E_k$ and $E_j$ that could lead to a sound concept of consistency. One might say that if the data represented by the matrix $B = [b^i_j]_{n\times n}$ are consistent then if the ratio of importance $E_i$ to $Ej$ is $\alpha_{ij}$, then for each triple $i, j, k$ we should have $\alpha_{ik}\alpha_{kj} = \alpha_{ij}$, and this could be used as basis for a formal definition of consistency.

We will say that an additive matrix $B = [b^i_j]_{n\times n}$ is *consistent*, if and only if, for all $i, j, k = 1, \ldots, n$, we have

$$\frac{b^i_k}{b^k_i} \cdot \frac{b^k_j}{b^j_k} = \frac{b^i_j}{b^j_i} \tag{6}$$

For any given multiplicative matrix $A = [a_{ij}]_{n\times n}$, let $B(A) = [b^i_j]_{n\times n}$ be derived from $A$ by $b^i_j = \frac{a_{ij}}{a_{ij}+1}$, and for any given additive matrix $B = [b^i_j]_{n\times n}$, let $A(B) = [a_{ij}]_{n\times n}$ be derived from $B$ by $a_{ij} = \frac{b^i_j}{1-b^i_j}$. Now we have $\frac{b^i_k}{b^k_i} \cdot \frac{b^k_j}{b^j_k} = \frac{b^i_k}{1-b^i_k} \cdot \frac{b^k_j}{1-b^k_j} = a_{ik}a_{kj} = \frac{b^i_j}{1-b^i_j} = a_{ij}$.

*Hence an additive matrix $B = [b^i_j]_{n\times n}$ is consistent if and only if the multiplicative matrix $A(B) = [a_{ij}]_{n\times n}$ is consistent.*

This provides an additional justification for the mappings $\phi, \phi^{-1}$ that transform $A$ into $B$ and $B$ into $A$ respectively. It also supports the combined pairwise comparisons process [6], [7], [16] that uses consistency index of $A(B)$ as a consistency index of $B$ without much explanation.

For both a multiplicative matrix $A = [a_{ij}]_{n\times n}$ and an additive matrix $B = [b^i_j]_{n\times n}$ the weights $w_1, \ldots, w_n$ are measures of importance of entities $E_1, \ldots, E_n$. Consider the entity $E_i$. All information about $E_i$ is stored in the sequence $a_{i1}, \ldots, a_{ik}$ - in case of matrix $A$, or $b^i_1, \ldots, b^i_n$ - in case of matrix $B$. For the matrix $A$, the weight corresponding to $E_i$ is defined as an eigenvalue $\lambda_i$ of $A$ [17], or a geometric mean $g_i = \sqrt[n]{a_{i1} \cdots a_{in}}$ [1]. When $A$ is consistent $a_{ij} = \frac{\lambda_i}{\lambda_j} = \frac{g_i}{g_j}$, which is one of the justifications of both methods [1], [17]. Now consider the sequence $b^i_1, \ldots, b^i_n$. While the value of $a_{ij}$ can be interpreted as 'absolute', due to $b^i_j + b^j_i = 1$, the value of $b^i_j$ is not 'absolute', it is 'relative to sum equal one'. The value of $\frac{b^i_j}{b^j_i}$ is 'absolute' so it can be used for weight calculation, but $\frac{b^i_j}{b^j_i} = a_{ij}$.

*This means the weights generated by $B$, similarly as for consistency, are the same as these generated but $A(B)$.*

This again provides some justification to the combined pairwise comparisons procedure [6], [7], [16].

Additive and multiplicative pairwise comparisons can be seen as orthogonal approaches to the same problem. This argument is briefly illustrated in Table II. When the difference between importance of $E_i$ and $E_j$ is small, $E_i \approx E_j$ looks as well justified, but still one of them seems to be slightly better, using additive pairwise comparisons, i.e. $b^i_j$, is superior to multiplicative approach. Dividing one hundred into, say, 53 to $E_i$ and 47 to $E_j$ is trustworthy and usually can have some justifications based on merits. On the other hand a statement like '$E_i$ is 1.13 times better than $E_j$', which is equivalent to '53 to 47' distribution of 100 points, can seldom be trusted or have convincing justification (unless as a derivation from $b^i_j$). Hence, when the initial qualitative judgment is $\approx$ or $\sqsupset$, but there is a reason to believe that we might be a little bit more precise, then the use of $b^i_j$ to represent qualitative relationship is superior to the use of $a_{ij}$. The situation seems to be opposite for the relations $>$ and $succ$. It is much easier to conclude that $E_i$ is about 5 times more important (better, etc.) than $E_j$ than to decide that the points distributions should be '83 to 17' (unless as a derivation from $a_{ij}$). We claim that when the initial qualitative judgment is $>$ or $\succ$, but there is a reason

TABLE II
SOME RELATIONSHIPS BETWEEN $a_{ij}$, $b^i_j$ AND $R_ij$ ($a_{ij} = \frac{b^i_j}{1-b^i_j}$ AND $b^i_j = \frac{a_{ij}}{a_{ij}+1}$).

| $b^j_i$ | $a_{ij}$ | $R_{ij}$ | $b^i_j$ | $a_{ij}$ | $R_{ij}$ | $a_{ij}$ | $b^k_i$ | $R_{ij}$ | $a_{ij}$ | $b^i_j$ | $R_{ij}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.5 | 1.0 | $\approx$ | 0.51 | 0.104 | $\approx$ | 2.0 | 0.67 | $\supset$ | 3.0 | 0.75 | $\supset$ |
| 0.52 | 1.08 | $\approx$ | 0.53 | 1.13 | $\approx$ | 4.0 | 0.8 | $>$ | 5.0 | 0.83 | $>$ |
| 0.54 | 1.17 | $\approx$ | 0.55 | 1.22 | $\approx$ | 6.0 | 0.86 | $>$ | 7.0 | 0.88 | $\succ$ |
| 0.56 | 1.27 | $\sqsupset$ | 0.57 | 1.33 | $\sqsupset$ | 8.0 | 0.89 | $\succ$ | 9.0 | 0.9 | $\succ$ |
| 0.58 | 1.38 | $\sqsupset$ | 0.59 | 1.44 | $\sqsupset$ | 10.00 | 0.91 | $\succ$ | | | |
| 0.60 | 1.5 | $\sqsupset$ | 0.63 | 1.7 | $\sqsupset$ | | | | | | |
| 0.66 | 1.94 | $\supset$ | 0.7 | 2.33 | $\supset$ | | | | | | |

to believe that we might be a little bit more precise, then the use of $a_{ij}$ to represent qualitative relationship is superior to the use of $b^i_j$. The relationship $\supset$ is a gray area, no approach seems to be superior to the other.

When the default values for multiplicative case are used, we recommend the default values 1 for the $\approx$ and $\sqsupset$ relationship and the default values 2 for $>$ and $\succ$ (gray cells in Table I). The Table III illustrate the proposed combined approach that involves multiplicative, additive and qualitative pairwise comparisons.

## V. DESCRIPTION OF PAIRWISE MATRIX INCONSISTENCY REDUCTION (PIXR)

To help with determining weights for attributes and parameters, pairwise inconsistency reduction is needed. The calculations and methods presented in this paper although simple and computable, may consume a lot of time and effort. Moreover, the number of calculations grows quadratically based on the number of parameters.

PiXR is an online tool that was developed to help with this computation and weight determination [14]. The tool consists of 4 main sections that guide the user through creating parameters, measurements, conversion between multiplicative & additive matrices, and inconsistency reduction. The tool also supports importing and exporting data as CSV files for ease of use.

PiXR operates on the following abbreviated process of inconsistency reduction and weight calculation [6]:

1) Pairwise matrix is provided to PiXR by Subject Matter Expert (SME)
   a) Matrix may be provided in as quantitative (additive or multiplicative) or qualitative pairwise matrix
   b) Consistency measure of the matrix is computed via equation 7
   c) Pairwise matrix is reduced using formulas 13 & 14 from [11]
   d) Consistency measure is then calculated again via equation 7, and SMEs can revise values and repeat the process
   e) Weights are calculated using the geometric mean of the columns of the reduced matrix

$$w_i = \sqrt[n]{\prod_{j=1}^{n} a_{ij}} \tag{7}$$

PiXR has 4 main sections:

1) *Parameters* – used for configuring names and order of parameters when populating matrices in later sections
2) *Parameter relations* (Qualitative) – used to configure relations between parameters using qualitative relations outlined in section III and Table I. Default values from Table I will be used to feed the next section. The user may also convert the qualitative matrix to multiplicative or additive.
3) *Quantitative Matrix* – an editable version of the matrix shown to the user and changes to it will be reflected in the computed consistency measure *cm* index. A reduction threshold in the range of 0.00-1.00 (inclusive), may be specified when reducing the matrix.
4) *Reduced Matrix and Weights* – displays the reduced matrix and the percentages/importance of the parameters. Reduced matrix and weights may be exported this section as CSVs.

PiXR comes with 3 sample applications/matrices from [6] which may be used to demonstrate the capabilities of the tool. The import/export function may be used to revise the matrices and repeat the matrix reduction process. SMEs may be involved in this iterative process to generate new weights and rankings.

PiXR is written in Typescript and Javascript XML (JSX). Using this tech stack enables the tool to run in the browser, thus saving users the time and effort needed to download and install binaries. One may argue that Typescript and by extension JavaScript to be slow for this kind of computation style since the algorithm is at worst $O(n^3)$. However, in most cases the number $n$ is relatively small. The approach is validated to converge quickly when using real world data [11] and by the theoretical results of [12].

PiXR aims to be *lazy* in most of its evaluations. When computing the triads to measure matrix consistency or computing the localized reduction value, patterns such as generators and streams are used where applicable. This lazy nature allows the user to experiment with the tool and provide as many features as they wish. The computation of the localized inconsistency reduction is also done in a lazy fashion as to not block the main thread and freeze the web-page.

Once the matrix is marked ready for reduction, it goes through the following steps:

1) Compute the current consistency measure

TABLE III
A SIMPLE EXAMPLE OF MODIFIED COMBINED PAIRWISE COMPARISONS PROCESS. IN $\mathsf{T}_2$ AND $\mathsf{T}_3$, GRAY CELLS CONTAIN $b^i_j$'S AND WHITE CELLS CONTAIN $a_{ij}$'S. IN $\mathcal{T}_4$, GRAY CELLS INDICATE DIFFERENCES FROM $\mathcal{T}_1$.

| $\mathcal{T}_1$ | $E_1$ | $E_2$ | $E_3$ | $E_4$ |
|---|---|---|---|---|
| $E_1$ | $\approx$ | $\sqsupset$ | $>$ | $\succ$ |
| $E_2$ | $\sqsubset$ | $\approx$ | $\supset$ | $\supset$ |
| $E_3$ | $<$ | $\subset$ | $\approx$ | $\approx$ |
| $E_4$ | $\prec$ | $\subset$ | $\approx$ | $\approx$ |

$\Longrightarrow$

| $\mathsf{T}_2$ | $E_1$ | $E_2$ | $E_3$ | $E_4$ |
|---|---|---|---|---|
| $E_1$ | 1.0 | 0.58 | 4.0 | 6.5 |
| $E_2$ | 0.42 | 1.0 | 2.0 | 3.0 |
| $E_3$ | 0.25 | 0.5 | 1.0 | 0.53 |
| $E_4$ | 0.154 | 0.333 | 0.47 | 1.0 |
| inconsistency $cm = 0.36 > 0.3$ | | | | |

$\Longrightarrow$

| $\mathsf{T}_3$ | $E_1$ | $E_2$ | $E_3$ | $E_4$ |
|---|---|---|---|---|
| $E_1$ | 1.0 | 0.6 | 4.1 | 6.0 |
| $E_2$ | 0.4 | 1.0 | 2.27 | 3.31 |
| $E_3$ | 0.244 | 0.44 | 1.0 | 0.548 |
| $E_4$ | 0.167 | 0.3 | 0.452 | 1.0 |
| $w_i$ | 8.2% | 13.5% | 33.5% | 44.7% |
| inconsistency $cm = 0.17 < 0.3$ | | | | |

$\Longrightarrow$

| $\mathcal{T}_4$ | $E_1$ | $E_2$ | $E_3$ | $E_4$ |
|---|---|---|---|---|
| $E_1$ | $\approx$ | $\sqsupset$ | $>$ | $>$ |
| $E_2$ | $\sqsubset$ | $\approx$ | $\supset$ | $>$ |
| $E_3$ | $<$ | $\subset$ | $\approx$ | $\approx$ |
| $E_4$ | $<$ | $<$ | $\approx$ | $\approx$ |

where $w_i = \sqrt[4]{a_{i1} \cdots a_{i4}}$

2) While the consistency measure is larger than threshold $\epsilon$, reduce the highest inconsistent triad $i, j, k$

The threshold $\epsilon$ has a default value of $0.3$ because it is an acceptable consistency measure with some mathematical justification [10], however it is usually domain dependent.

The above algorithm in pseudo-code is Algorithm 1, where $consistencyMeasure()$ is computed using equation 7. Algorithm 2 is the algorithm for reducing the most inconsistent triad. It is based on the results of [11].

**Algorithm 1** Inconsistency Index $cm$

```
cm ← consistencyMeasure(matrix)
while cm ≥ ϵ do
    reduceTriad(matrix)
    cm ← consistencyMeasure(matrix)
end while
```

**Algorithm 2** Most Inconsistent Triad

```
ij ← matrix[i][j]
ik ← matrix[i][k]
kj ← matrix[k][j]
a ← ik * kj {Following equation 13 & 14 from [11]}
factor ← 1
if a > ij then
    factor ← −1
end if
b ← factor * (ij + 2 * a)
c ← a − ij
Δ_c ← min(computeRoots(a, b, c)) {least positive root}
matrix[i][k] ← ik + (factor * ik * Δ_c)
matrix[k][j] ← kj + (factor * kj * Δ_c)
matrix[i][j] ← ij − (factor * ij * Δ_c)
```

## VI. FINAL COMMENT

A relationships between multiplicative, additive and qualitative pairwise comparisons have been discussed in detail. It was shown that from purely mathematical point of view multiplicative and additive pairwise comparisons are equivalent, but for applications, dependently on the range of data and qualitative relations, one approach can be superior to another. Altogether, a combined approach is recommended. The on-line tool, PiXR, is also provided. The fundamental problem with all methods like the one presented here, is trust. We believe our approach is more trustworthy than the standard AHP, mainly because we better address the problem of assigning numerical values to subjective qualitative judgements.

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