Communication Papers of the 18th Conference on Computer Science and Intelligence Systems pp. 75–80
DOI: 10.15439/2023F8192
ISSN 2300-5963 ACSIS, Vol. 37

Star-critical Ramsey numbers for hexagon

Tomasz Dzido
0000-0001-6712-8541
Department of Information Systems
Gdynia Maritime University
ul. Morska 81/87, 81-225 Gdynia, Poland
Email: t.dzido@wznj.umg.edu.pl

***Abstract*—Erdös and Faudree stated that it is an interesting problem to determine all the graph pairs which are Ramsey-full. For even cycles, they only showed that the pair (C_4, C_4) is Ramsey-full. It turns out that this statement cannot be applied to longer even cycles. Wu, Sun and Radziszowski obtained that the pair (C_n, C_4) for $n > 4$ is not Ramsey-full. In this article we will show that the pairs (C_n, C_6) for different values of n are also not Ramsey-full.**

We will also determine the values of some star-critical Ramsey numbers, in particular $r_*(C_6, C_6) = 6$ and $r_*(C_7, C_6) = 7$. In addition, we also show other values and bounds for star-critical Ramsey numbers for two cycles, one of which is an even cycle. These results are the beginning of the star-critical Ramsey number problem for even cycles of length 6 or more, and may help in obtaining further properties of this type.

I. Introduction

THE theorem, later called Ramsey's theorem, was proved by Ramsey and published shortly after his death in 1930. Informally written, this theorem proves that "complete disorder is impossible". In other words, any sufficiently large structure contains a substructure with the desired property. One of the popular ways looking at Ramsey's theory is in the context of graph theory, and more specifically edge coloring of graphs. To put it quite simply, we want to answer the following question: If we have a complete graph K_n on n vertices where every edge is arbitrarily colored either blue or red, what is the smallest value of n that guarantees the existence of either a subgraph G_1 which is red, or a subgraph G_2 which is blue? This smallest search n is called a 2-color Ramsey number $R(G_1, G_2)$. Initially, only the case when subgraphs G_1 and G_2 are complete subgraphs was considered. Therefore, Ramsey numbers for subgraphs other than complete and those defined analogously for more subgraphs and colors became popular very quickly. Currently, many classes of graphs are considered, such as paths, stars or cycles considered in this article.

From the informal definition of Ramsey numbers presented above, it follows that there is a critical graph, i.e. an edge coloring of a complete graph of order $n-1$, which does not contain a red copy of G_1 or a blue copy of G_2. Therefore, each 2-edge coloring of K_n contains either red G_1 or blue G_2, and there is a coloring of K_{n-1} without red G_1 or blue G_2. These facts lead us to an interesting question. For known Ramsey numbers, $R(G_1, G_2)$ = n, and a 2-coloring of the graph $K_{n-1} + v$, if we add colored edges individually from a new vertex v to vertices of K_{n-1}, then at what point must the graph have a red G_1 or a blue G_2? Alternatively, what is the largest star that can be removed from K_n so that the underlying graph is still forced to have either a red G_1 or a blue G_2? To study this, Hook and Isaak [6] introduced the definition of the star-critical Ramsey number $r_*(G_1, G_2)$.

Numerous other varieties of non-classical Ramsey numbers have been defined. For example: bipartite, planar, on-line, induced, local, diagonal, geometric, rainbow, linear and star-critical that are considered in this work. Many interesting applications of Ramsey theory arose in the field of mathematics and computer science, these include results in number theory, algebra, geometry, topology, set theory, logic, information theory and theoretical computer science. The theory is especially useful in building and analyzing communication nets of various types. Ramsey theory has been applied by Frederickson and Lynch to a problem in distributed computations [5], and by Snir [12] to search sorted tables in different parallel computation models. The reader will find more applications in Rosta's summary titled "Ramsey Theory Applications" [11].

II. Definitions and known results

In this paper we consider only finite and simple graphs. Let $G = (V(G), E(G))$. The deletion of edges of a copy of a subgraph H from G will be denoted as $G - H$ and the deletion of an edge e from G will be denoted as $G - e$. Let K_n denote a complete graph on n vertices and $K_{m,n}$ a complete bipartite graph on $m + n$ vertices. Denote by C_n a cycle of order n.

Definition 1. *The circumference $c(G)$ of a graph G is the length of its longest cycle.*

Definition 2. *The girth $g(G)$ of a graph G is the length of its shortest cycle.*

Definition 3. *A graph is called weakly pancyclic if it contains cycles of every length between the girth and the circumference.*

The following terminology, definitions and some descriptions are taken from [16].

Definition 4. *Given two graphs G_1 and G_2, we say that a graph G arrows the pair (G_1, G_2), denoted by $G \to (G_1, G_2)$, if in any red/blue coloring of the edges of G, there is a red copy of G_1 or a blue copy of G_2.*

For two given graphs G_1 and G_2, the most extensively investigated concept within Ramsey theory is *the graph Ramsey*

Thematic track: Complex Networks – Theory and Application

number $R(G_1, G_2)$, which is the smallest integer r such that, for any graph G of order r, either G contains G_1 as a subgraph or G contains G_2 as a subgraph, where $\overline{G}$ is the complement of G. For simplicity, we now restate this definition of $R(G_1, G_2)$ in the language of arrowing.

Definition 5. $r = R(G_1, G_2) = min\{n | K_n \to (G_1, G_2)\}$.

Let r denote the Ramsey number $R(G_1, G_2)$ throughout the paper. A dynamic survey on Ramsey numbers can be found in [10].

Since $K_r \to (G_1, G_2)$, but $K_{r-1} \nrightarrow (G_1, G_2)$, a natural problem is to consider G such that $K_{r-1} \subseteq G \subseteq K_r$ and $G \to (G_1, G_2)$. To study this, Hook and Isaak [6] introduced the definition of *the star-critical Ramsey number* $r_*(G_1, G_2)$.

Definition 6 ([6]). $r_*(G_1, G_2) = min\{k | K_{r-1} \sqcup K_{1,k} \to (G_1, G_2)\}$.

The values of many star-critical Ramsey numbers have been determined. We will only recall the results for two cycles. In [16], Zhang, Broersma and Chen showed the following results.

Theorem 7 ([16]). $r_*(C_n, C_m) \geq \frac{m}{2} + 3$ *for even* $m \geq 4$, *odd* $n \geq \frac{3m}{2}$, *and for even* $m \geq 4$, *even* $n \geq m$, $n \geq 6$.

Theorem 8 ([16]). *For* m *odd,* $n \geq m \geq 3$ *and* $(m, n) \neq (3, 3)$, $r_*(C_n, C_m) = n + 1$.

Wu, Sun and Radziszowski [14] obtained that $r_*(C_n, C_4) = 5$ for $n \geq 4$. This result indicates that star-critical Ramsey number can be constant and much smaller than the corresponding classical Ramsey number. A fairly extensive and interesting summary of the all known results for star-critical Ramsey numbers can be found in the article [9]. One of the open problems appearing in various articles is the determination of the values of the numbers $r_*(C_n, C_m)$ for even m and $n \geq m \geq 6$. In this article, we focus on cycle C_6 and present a number of new values and bounds. In particular, we determine the following results: $r_*(C_6, C_6) = 6$ and $r_*(C_7, C_6) = 7$.

In the context of $G \to (G_1, G_2)$ and star-critical Ramsey numbers, some other definition was introduced.

Definition 9 ([16]). *A pair of graphs* (G_1, G_2) *is called Ramsey-full if* $K_r \to (G_1, G_2)$, *but* $K_r - e \nrightarrow (G_1, G_2)$.

Erdös and Faudree [3] stated that it is an interesting problem to determine all the graph pairs which are Ramsey-full. All the known graph pairs which are Ramsey-full are summarized in [16]. In the case of two cycles, we know that the pair (C_4, C_4) is Ramsey-full [3]. Wu, Sun and Radziszowski [14] obtained that the pair (C_n, C_4) for $n > 4$ is not Ramsey-full. The same is true for larger even cycles, as evidenced by the results obtained in this article for star-critical Ramsey numbers. In this article we will show that the pairs (C_n, C_6) for different values of n are also not Ramsey-full.

III. Preliminary results

The following notation and terminology comes from [2].

For positive integers a and b we define $r(a, b)$ as

$$r(a, b) = a - b\lfloor \frac{a}{b} \rfloor = a \bmod b.$$

For integers $n \geq k \geq 3$, we define $w(n, k)$ as

$$w(n, k) = \frac{1}{2}(n-1)k - \frac{1}{2}r(k - r - 1),$$

where $r = r(n - 1, k - 1)$.

Woodall's theorem [13] can then be written as follows.

Theorem 10 ([2]). *Let* G *be a graph on* n *vertices and* m *edges with* $m \geq n$ *and* $c(G) = k$. *Then*

$$m \leq w(n, k)$$

and this result is the best possible.

Lemma 11 ([1]). *Every nonbipartite graph* G *of order* n *with* $|E(G)| > (n-1)^2/4 + 1$ *is weakly pancyclic with* $g(G) = 3$.

For a graph G, define the Turán number $ex(n, G)$ to be the largest integer m such that there exists a graph on n vertices with m edges that does not contain G as a subgraph. In other words, if H has n vertices and more than $ex(n, G)$ edges, then H must contain G as a subgraph. A graph on n vertices is said to be *extremal with respect to G* if it does not contain a subgraph isomorphic to G and has exactly $ex(n, G)$ edges.

It is easy to see that for odd cycles, the Turán number $ex(n, C_{2t+1}) = \lfloor \frac{n^2}{4} \rfloor$ for $n > 4t - 1$, since no bipartite graph contains an odd cycle. For smaller values of n, we also know the value of $ex(n, C_{2t+1})$. Write n in the form $n = (s-1)(2t-1) + r$ where $s \geq 1$, $2 \leq r \leq 2t$ are integers. Then we have the following property.

Theorem 12 ([4]). *For any* $n \geq 1$ *and* $2t + 1 \geq 5$,

$$ex(n, C_{2t+1}) = (s-1)\binom{2t}{2} + \binom{r}{2}, \text{ for } 2t+1 \leq n \leq 4t-1.$$

However, the problem of determining the Turán numbers for even cycles is still open. In the case of cycle C_6, we know all values of $ex(n, C_6)$ for $n < 22$ and all exstremal graphs with respect to C_6 for these numbers. These results are included in the paper [15].

We define the bipartite Turán number $ex(m, n, H)$ of a graph H to be the maximum number of edges in an H-free bipartite graph with parts of sizes m and n.

Theorem 13 ([8]). *Let* t *be an integer and* $G = (X, Y; E)$ *be a bipartite graph. Suppose* $|X| = n$, $|Y| = m$, *where* $n \geq m \geq t \geq \frac{m}{2} + 1$. *Then*

$$ex(m, n, C_{2t}) = (t-1)n + m - t + 1.$$

IV. RESULTS

When $n \neq 4$ is even, $r = R(C_n, C_n) = \frac{3n}{2} - 1$. A new proof of this classic result was given by Károlyi and Rosta [7]. Erdös and Faudree [3] showed that (C_4, C_4) is Ramsey-full. It turns out that this is not the case for longer cycles of even length.

Theorem 14. *$K_r - e \to (C_6, C_6)$, where $r = R(C_6, C_6) = 8$.*

Proof. Let G be a graph $K_8 - e$ with $|V(G)| = 8$ and $|E(G)| = \binom{8}{2} - 1$. Let us consider an arbitrary coloring all the edges of the graph G. For any red/blue edge coloring of G, let G^R (G^B) be the graph whose vertex set is $V(G)$ and edge set consists of all red (blue) edges of G, respectively. Suppose to the contrary that neither G^R nor G^B contains a C_6. Since $|E(G)| = \binom{8}{2} - 1 = 27$, then without loss of generality we can assume that $|E(G^R)| \geq \lceil \frac{\binom{8}{2}-1}{2} \rceil = 14$. By Lemma 11, G^R is weakly pancyclic with girth 3. On the other hand, $w(8, 4) = 13$ and by Theorem 10, G^R contains a C_5 as a subgraph.

Claim 14.1. *If G^B contains a monochromatic C_7, then G^R contains a monochromatic C_6.*

Proof. Let $C = x_1x_2...x_7x_1$ be a blue C_7 in G. Were some 2-chord of C blue, G would contain a blue C_6, a contradiction. Whence all 2-chords of C are red. In particular, the 2-chords of C form a red C_7 with at most one edge deleted (we consider $K_8 - e$). First, if we have a red C_7, then by pancyclicity of G^r we immediately have a red C_6. Assume we have a red $C_7 - e = x_1x_3x_5x_7x_2x_4x_6$. In order to avoid a red C_6, x_1x_4 and x_3x_6 are blue. Thus $x_1x_4x_5x_6x_3x_2x_1$ is a blue C_6, a contradiction. □

Let $C = v_1v_2v_3v_4v_5v_1$ be a cycle of length 5 in G^R. Let $C^* = \{w_1, w_2, w_3\}$ denote the set of vertices of G not in C. To avoid a red C_6, every vertex C^* is red incident to at most two vertices in C.

Claim 14.2. *If there are two red edges connecting $v \in C^*$ and C, say wv_i and wv_j, then we have $|i - j| = 2$ or 3.*

Proof. If the above condition does not hold, then it is easy to see that G contains a red C_6, a contradiction. □

Observe that the possible pairs of vertices in C that can be joined to vertex $w_i \in C^*$ are $P = \{v_1v_3, v_1v_4, v_2v_4, v_2v_5, v_3v_5\}$. Let $A_i = \{v \in C | vw_i \in G^R\}$ where $i \in \{1, 2, 3\}$.

Claim 14.3. *Given two integers $i, j \in \{1, 2, 3\}$, if $|A_i| = |A_j| = 2$ and $A_i \cap A_j = \emptyset$, then G^R contains a red C_6.*

Proof. Without los of generality, let us assume that the set $D = \{w_1v_1,\ w_1v_3,\ w_2v_2,\ w_2v_4\}$ is the set of red edges connecting the vertices w_1, w_2 with the cycle C. Then we immediately have a red $C_7 = w_1v_3v_2w_2v_4v_5v_1w_1$ and by pancyclicity of G^R, we have a red C_6. □

The rest of the proof contains all possible cases of setting the maximum number of red edges between C and C^*. Therefore, we want to consider all possible maximal structures of A_i and show that we always get a monochromatic cycle C_6. We start from the case where all vertices w_i are connected by red edges to the same vertices from cycle C (Claim 14.4). Later we consider the case where two of A_i have the same structure and the third one has a different structure (Claim 14.5). Finally, we show what other cases remain (Claim 14.6) and consider them (Claims 14.7 and 14.8). Keep in mind that we are dealing with $K_8 - e$. This means that it may happen that one of the red edges connecting C and C^* may not be there.

Claim 14.4. *For each $i \in \{1, 2, 3\}$, let $A_i \subseteq \{v_m, v_n\}$ with $v_mv_n \in P$. Then G^B contains a monochromatic C_6.*

Proof. Without los of generality, let us assume that $A_i \subseteq \{v_1, v_3\}$ for each $i \in \{1, 2, 3\}$. Consider now the blue bipartite subgraph F with parts $\{v_2, v_4, v_5\}$ and $\{w_1, w_2, w_3\}$. Then $|E(F)| \geq 8$ (we consider $K_8 - e$) $> ex(3, 3, C_6) = 7$, according to Theorem 13. □

Without loss of generality, consider the case where $A_i \subseteq \{v_1, v_3\}$ for $i \in \{1, 2\}$. Note that in this situation $A_3 \subseteq \{v_2, v_4\}$ or $A_3 \subseteq \{v_1, v_4\}$. The case $A_3 \subseteq \{v_2, v_5\}$ is the same as the first variant, and the case $A_3 \subseteq \{v_3, v_5\}$ is the same as the second.

Claim 14.5. *Let $A_i \subseteq \{v_1, v_3\}$ for $i \in \{1, 2\}$ and $A_3 \subseteq \{v_2, v_4\}$ or $A_3 \subseteq \{v_1, v_4\}$. Then G contains a monochromatic C_6.*

Proof. 1) $A_i \subseteq \{v_1, v_3\}$ for $i \in \{1, 2\}$ and $A_3 \subseteq \{v_2, v_4\}$. Let us consider all blue edges connecting C and C^* and all possible edges from A_i that form the set $R = \{w_1v_1, w_1v_3, w_2v_1, w_2v_3, w_3v_2, w_3v_4\}$. Consider the case where one of the edges in R does not exist in $G = K_8 - e$. Note that every edge in R if it is blue, then it is part of some blue C_6 in the bipartite graph $[C, C^*]$. Taking into account this fact and the thesis of Claim 3, without loss of generality, we can consider a situation where there is no edge w_3v_2. This means that the edges w_1v_1, w_1v_3, w_3v_4 are colored red, then in order to avoid red C_6, the edges v_2v_4, v_2v_5, w_1w_3 edges are colored blue. We then get the blue cycle $w_2v_2v_4w_1w_3v_5w_2$.
It remains to consider a situation in which there is no edge belonging to the rest (without edges from the set R) of the bipartite graph $[C, C^*]$. As a result of the analysis of the structure of the sets A_i, without loss of generality, we obtain the following 3 cases.

a) There is no edge w_1v_2 in graph G.
In order to avoid the following blue 6-cycles: $w_1v_3w_3v_5w_2v_4w_1$, $w_1v_1w_3v_5w_2v_4w_1$, the edges w_1v_3, w_1v_1 must be colored red. Then the edges v_2v_4 and v_2v_5 must be colored blue. Note that then the edge w_1w_2 must be blue. Suppose, on the contrary, that w_1w_2 is red. In this case, edges w_2v_1 and w_2v_3 must be blue and we get a blue 7-cycle: $w_3v_1(v_3)w_2v_2v_4w_1v_5w_3$, and by Claim 1 we get a red cycle C_6. To avoid the next two blue

cycles $w_3v_2v_4w_1w_2v_5w_3$ and $w_3v_4v_2w_2w_1v_5w_3$, edges w_3v_2 and w_3v_4 must be colored red. But then by Claim 3 we have a red cycle C_6.

b) There is no edge w_1v_5 in graph G.
In order to avoid the following blue 6-cycles: $w_2v_2w_1v_4w_3v_5w_2$, $w_3v_2w_1v_4w_2v_5w_3$, $w_1v_1w_3v_5w_2v_2w_1$ and $w_1v_3w_3v_5w_2v_2v_1$, the edges w_3v_4, w_3v_2, w_1v_1 and w_1v_3 are colored red. By Claim 3 we immediately obtain a red cycle of length 6.

c) There is no edge w_3v_1 in graph G.
In order to avoid the following blue 6-cycles: $w_2v_2w_1v_4w_3v_5w_2$, $w_3v_2w_1v_4w_2v_5w_3$ and $v_2w_2v_5w_3v_3w_1v_2$, the edges w_1v_3, w_3v_2 and w_3v_4 are red. Then the edges v_3v_5 and w_1w_3 must be blue and we have the blue 6-cycle: $v_3v_5w_2v_4w_1w_3v_3$.

2) $A_i \subseteq \{v_1, v_3\}$ for $i \in \{1, 2\}$ and $A_3 \subseteq \{v_1, v_4\}$.
Consider the blue bipartite subgraph F with parts $\{v_2, v_4, v_5\}$ and $\{w_1, w_2, w_3\}$. Then $|E(F)| \geq 8 > ex(3, 3, C_6) = 7$, according to Theorem 13. This means that only some edge of subgraph F may be missing in G. As a result of the analysis of the structure of the blue 6-cycles in F, without loss of generality, we obtain the following 2 cases.

a) There is no edge w_1v_4 in graph G
First, let's note that to avoid the blue cycle C_6, the edges w_1v_3, w_2v_3 and w_3v_4 must be colored red. Then consider the edges connecting vertex v_1 with vertices from C^*. At least two of them must be red. Suppose the edge w_1v_1 is colored red. Then the edges v_2v_4, v_2v_5, w_1w_2, w_1w_3 and w_2w_3 must be blue. We obtain the following blue 6-cycle: $w_1v_5v_2v_4w_2w_3w_1$. Finally, let us consider the case when the edge w_1v_1 is colored blue. This leads to the fact that both edges w_2v_1 and w_3v_1 are red, while the edges v_2v_4, v_2v_5, v_3v_5 and w_2w_3 are blue. Hence we have a blue cycle of length 6: $w_2v_4v_2v_5v_3w_3w_2$.

b) There is no edge w_3v_2 in graph G
The proof is identical to that of subcase (a). □

Let us now summarize the above cases and indicate which ones still remain to be considered.

Claim 14.6. *Without loss of generality, the maximum possible structures of A_i can be:*

1) $A_1 \subseteq \{v_1, v_3\}$, $A_2 \subseteq \{v_1, v_3\}$ *and* $A_3 \subseteq \{v_1, v_3\}$
2) $A_1 \subseteq \{v_1, v_3\}$, $A_2 \subseteq \{v_1, v_3\}$ *and* $A_3 \subseteq \{v_2, v_4\}$
3) $A_1 \subseteq \{v_1, v_3\}$, $A_2 \subseteq \{v_1, v_3\}$ *and* $A_3 \subseteq \{v_1, v_4\}$
4) $A_1 \subseteq \{v_1, v_3\}$, $A_2 \subseteq \{v_2, v_4\}$ *and* $A_3 \subseteq \{v_1, v_4\}$
5) $A_1 \subseteq \{v_1, v_3\}$, $A_2 \subseteq \{v_2, v_4\}$ *and* $A_3 \subseteq \{v_2, v_5\}$

Proof. Cases 1-3 have already been considered above. It remains to prove that cases 4-5 exhaust the situation when all sets A_i can be different. For this problem, let us consider situations where $A_1 \subseteq \{v_1, v_3\}$ and $A_2 \subseteq \{v_2, v_4\}$ or $A_2 \subseteq \{v_1, v_4\}$. Note that the case $A_2 \subseteq \{v_2, v_5\}$ is the same as the first variant, and the case $A_2 \subseteq \{v_3, v_5\}$ is the same as the second. For both variants let us analyze all possible maximal structures of A_3 and notice that all possible structures fall into cases 2-5.

1) $A_1 \subseteq \{v_1, v_3\}$ and $A_2 \subseteq \{v_2, v_4\}$
   a) $A_3 \subseteq \{v_1, v_3\}$ - Case 2
   b) $A_3 \subseteq \{v_1, v_4\}$ - Case 4
   c) $A_3 \subseteq \{v_2, v_4\}$ - Case 2
   d) $A_3 \subseteq \{v_2, v_5\}$ - Case 5
   e) $A_3 \subseteq \{v_3, v_5\}$ - Case 5
2) $A_1 \subseteq \{v_1, v_3\}$ and $A_2 \subseteq \{v_1, v_4\}$
   a) $A_3 \subseteq \{v_1, v_3\}$ - Case 3
   b) $A_3 \subseteq \{v_1, v_4\}$ - Case 3
   c) $A_3 \subseteq \{v_2, v_4\}$ - Case 4
   d) $A_3 \subseteq \{v_2, v_5\}$ - Case 5
   e) $A_3 \subseteq \{v_3, v_5\}$ - Case 4 □

Claim 14.7. *Let $A_1 \subseteq \{v_1, v_3\}$, $A_2 \subseteq \{v_2, v_4\}$ and $A_3 \subseteq \{v_1, v_4\}$. Then G contains a monochromatic C_6.*

Proof. Consider the blue bipartite subgraph with parts C and C^*. Note that this subgraph contains the following cycle of length 6: $v_2w_1v_5w_2v_3w_3v_2$. This means that only some edge of this cycle may be missing in graph G. We obtain the following 6 cases.

1) There is no edge w_3v_2 in graph G
In order to avoid the following blue 6-cycles: $w_1v_2w_2v_3w_3v_5w_1$, $w_1v_4w_2v_3w_3v_5w_1$, $w_1v_1w_2v_3 - w_3v_5w_1$, the edges w_2v_2, w_2v_4 and w_1v_1 must be colored red. Then the edges v_1v_3, v_3v_5 and w_1w_2 must be colored blue. If the edge w_1v_3 is blue, we obtain the following blue cycle: $w_3v_3v_1w_2w_1v_5w_3$. This means that the edge w_1v_3 is red. But then, based on Claim 3, we have a red cycle C_6.
2) There is no edge w_1v_5 in graph G
In a similar way as in the previous case, in order to avoid the following blue 6-cycles: $w_1v_2w_3v_3w_2v_1w_1$, $w_1v_2w_3v_5w_2v_3w_1$, $w_1v_4w_2v_3w_3v_2w_1$, the edges w_1v_1, w_1v_3 and w_2v_4 must be colored red. Then the edges v_2v_4, v_2v_5 and w_1w_2 must be colored blue. We have the blue 7-cycle: $w_1v_4v_2v_5w_3v_3w_2w_1$, and by Claim 1 we obtain a red cycle C_6.
3) There is no edge w_1v_2 in graph G
As before, to avoid the following blue 6-cycles: $w_1v_1w_2v_3w_3v_5w_1$, $w_2v_4w_1v_5w_3v_3w_2$, the edges w_1v_1 and w_2v_4 must be colored red. If edge w_1v_3 is colored red then similarly to case 2 we have the blue 7-cycle: $w_1v_4v_2v_5w_3v_3w_2w_1$, and by Claim 1 we obtain a red cycle C_6. If the edge w_2v_2 is red, then, as in case 1, we obtain the following blue cycle: $w_3v_3v_1w_2w_1v_5w_3$. If both edges w_1v_3 and w_2v_2 are blue, we have the blue 6-cycle: $w_2v_2w_3v_3w_1v_5w_2$. This means that these two edges are red. But then, based on Claim 3, we have a red 6-cycle.
4) There is no edge w_3v_3 in graph G

The proof of this case is analogous to the proof of previous cases. We start by noting that because of the cycles $w_1v_4w_2v_5w_3v_2w_1$, $w_1v_1w_2v_5w_3v_2w_1$, $w_1v_3w_2v_5w_3v_2w_1$, the edges w_2v_4, w_1v_1 and w_1v_3 are red. The rest of the reasoning is as in the above cases.

5) There is no edge w_2v_5 in graph G
The proof of this case is almost identical to the proof of case 3, so we leave it to the reader.
6) There is no edge w_2v_3 in graph G
Avoiding the corresponding blue cycles of length 6, we get that the edges w_1v_1 and w_2v_4 must be colored red. Then consider the possible colors of edges w_1v_3 and w_2v_2. If both of these edges are red, then by Lemma 3, we immediately have a red 6-cycle. If both are blue, we have the following blue cycle: $w_1v_3w_3v_5w_2v_2w_1$. The situation remains when both of these edges have different colors. It is similar to the situations considered in cases 1-3, so we omit it. □

Claim 14.8. *Let $A_1 \subseteq \{v_1, v_3\}$, $A_2 \subseteq \{v_2, v_4\}$ and $A_3 \subseteq \{v_2, v_5\}$. Then G contains a monochromatic C_6.*

Proof. First, note that the blue bipartite graph with partitions C and C^* contains the following two 6-cycles: $v_4w_1v_5w_2v_1w_3v_4$ and $v_3w_2v_5w_1v_4w_3v_3$. This means that only an edge occurring in both of these cycles can be missing in G. Due to this fact, we are given the following 4 cases to consider.

1) There is no edge w_1v_4 in graph G
To avoid the following blue cycles $w_1v_5w_2v_3w_3v_1w_1$ and $w_1v_3w_3v_1w_2v_5w_1$, the edges w_1v_1 and w_1v_3 must be colored red. Then the edges v_2v_5 and v_2v_4 are blue. We have the following blue 7-cycle: $w_1v_2v_4w_3v_1w_2v_5w_1$. Taking into account the thesis of Claim 1, we obtain a red cycle of length 6.
2) There is no edge w_3v_4 in graph G
Again considering the same cycles as at the beginning of the proof of case 1, we have that edges w_1v_1 and w_1v_3 are red. In order to avoid the blue 6-cycle $v_2w_3v_3w_2v_5w_1v_2$, the edge w_3v_2 will also be red. This forces edges v_2v_4, v_2v_5 and w_1w_3 to be colored blue. This all leads us to the blue cycle of length 6: $w_1w_3v_1w_2v_5v_2w_1$.
3) There is no edge w_1v_5 in graph G
As in the previous two cases, we get that the edges w_1v_1, w_1v_3 and w_2v_2 are red, while the edges v_2v_4, v_2v_5 and w_1w_2 are blue. Taking this into account, we immediately obtain the blue cycle of length 6: $w_2v_5v_2v_4w_3v_3w_2$.
4) There is no edge w_2v_5 in graph G
In this case we obtain the same red and blue edges as at the beginning of the proof of case 3. This time we have the following blue cycle of length 7: $w_1v_5v_2v_4w_3v_3w_2w_1$. From Claim 1 we also have a red cycle of length 6. □

We have already considered all possible cases and in each of them we have obtained a monochromatic C_6, so the proof of the theorem is complete. □

Corollary 15. *The pair of graphs (C_6, C_6) is not Ramsey-full.*

Corollary 16.

$$r_*(C_6, C_6) = 6.$$

Proof. We know that $R(C_6, C_6) = 8$ [7]. The lower bound follows easily from Theorem 7 in the special case $n = m = 6$. The upper bound follows directly from the conclusion of above Theorem 14. □

Theorem 17. *For even $m \geq 6$, odd $k \geq 1$ and $k \leq \frac{m}{2}$, $r_*(C_{m+k}, C_m) \geq m + 1$.*

Proof. Since $m+k$ is odd, then $r = R(C_{m+k}, C_m) = 2m-1$ [7]. Let P_1, P_2 be a partition of $V(K_{r-1})$ with $|P_1| = |P_2| = m - 1$. Assign colors to the edges of the K_{r-1} as follows: color the edges of P_1 and P_2 blue and all the other edges red. Let p_0 be an additional vertex, which is adjacent to P_1 with $m-1$ red edges and adjacent to P_2 with one blue edge. It is easy to check that there is neither a red C_{m+k} nor a blue C_m. □

By $K_{p1} * K_{p2} * ... * K_{pi}$ we denote a blockgraph, which consists of i complete blocks $K_{p1}, ..., K_{pi}$ such that exactly one vertex is contained in any of these complete subgraphs. Using this notation, for three graphs G, H and I, the graph $G * (H * I)$ consists of two graphs G and $H * I$, which have exactly one common vertex which is contained in G and H.

Theorem 18.

$$r_*(C_7, C_6) = 7.$$

Proof. From [7] we know that $R(C_7, C_6) = 11$. In oder to determine the value of $r_*(C_7, C_6)$, it is enough to prove that $F = K_{11} - K_{1,3} \rightarrow (C_7, C_6)$ because the lower bound follows from Theorem 17. Let us consider an arbitrary red/blue coloring all the edges of the graph F. For this coloring of F, let F^R (F^B) be the graph whose vertex set is $V(F)$ and edge set consists of all red (blue) edges of F, respectively. Suppose to the contrary that F^R does not contain a C_7 and F^B does not contain a C_6. The following results are taken from papers [15] and [4], respectively.

Claim 18.1 ([15])**.** *$ex(11, C_6) = 23$ and there are exactly three extremal graphs with respect to C_6 for this number, namely $K_5 * K_3 * K_5$, $K_5 * (K_3 * K_5)$ and $K_5 * (K_5 * K_3)$.*

Claim 18.2 ([4])**.** *$ex(11, C_7) = 30$ and there are exactly two extremal graphs with respect to C_7 for this number, namely $K_6 * K_6$ and $K_{5,6}$.*

Since $|E(F)| = 52$, then $|E(F^B)| \geq 23$ or $|E(F^R)| \geq 30$. Note that in the complement of each of the extremal graphs with respect to C_6 or C_7 for these numbers, we obtain a red C_7 or a blue C_6, respectively. We have a contradiction, which completes the proof of the theorem. □

Again, using the results of [15] and [4], we can easily obtain the following theorem.

Theorem 19. *$K_r - e \to (C_k, C_6)$, where $r = R(C_k, C_6)$ and $k \in \{17, 19\}$.*

Proof. Based on the works [7], [15] and [4] we have $R(C_{17}, C_6) = 19$, $R(C_{19}, C_6) = 21$, $ex(19, C_{17}) = 126$, $ex(19, C_6) = 44$, $ex(21, C_{19}) = 159$ and $ex(21, C_6) = 50$. Note that for $k \in \{17, 19\}$ the property $ex(r, C_k) + ex(r, C_6) = |E(K_r)| - 1$ holds. With a simple analysis the complements of critical graphs with respect to C_k described in [15], we have the proof. □

For graphs G_1, G_2 a coloring f is a $(G_1, G_2; n)-coloring$ if f is a red/blue edge coloring all the edges of K_n and f contains neither a red G_1 nor a blue G_2. A coloring $(G_1, G_2; n)$ is said to be *critical* if $n = R(G_1, G_2) - 1$.

Two more results can be obtained by simple computer methods.

Theorem 20.

$$r_*(C_8, C_6) = 6,$$
$$r_*(C_9, C_6) = 7.$$

Proof. On the website *https://users.cecs.anu.edu.au/∼bdm/data /graphs.html* we can find a database of all non-isomorphic graphs of order up to 11. They can be easily filtered out, yielding 24 critical colorings $(C_8, C_6; 9)$ for $r_1 = R(C_8, C_6) = 10$ and 26 critical colorings $(C_9, C_6; 10)$ for $r_2 = R(C_9, C_6) = 11$. Then we take all these critical colorings and consider all possible colorings of type $K_{r_1 - 1} \sqcup K_{1,k}$ and $K_{r_2 - 1} \sqcup K_{1,k}$ for increasing values of k, starting from $k = 1$. We are looking for the largest value of k that there is a coloring without forbidden subgraphs. □

Let's end the article with two interesting questions.

Question 1. *Let us note that $r_*(C_6, C_6) = r_*(C_8, C_6) = 6$ and $r_*(C_7, C_6) = r_*(C_9, C_6) = 7$. Will it turn out that $r_*(C_n, C_6) = 6$ or 7 depending on the parity of n?*

Question 2. *Do similar relationships hold for even cycles longer than 6?*

References

[1] S. Brandt, "A sufficient condition for all short cycles," *Discrete Appl. Math.,* vol. 79, 1997, pp. 63–66.

[2] L. Caccetta, K. Vijayan, "Maximal cycles in graphs," *Disc. Math.,* vol. 98, 1991, pp. 1–7.

[3] P. Erdös, R. J. Faudree, "Size Ramsey functions," *Sets, Graphs and Numbers,* 1991, pp. 219–238.

[4] Z. Füredi, D. S. Gunderson, "Extremal Numbers for Odd Cycles," *Combinatorics, Probability and Computing,* vol. 24(4), 2014, pp. 641–645.

[5] G. Frederickson, N. Lynch, "Electing a leader in a synchronous ring," *Journal of Association for Computing Machinery,* vol. 34, 1987, pp. 98–115.

[6] J. Hook, G. Isaak, "Star-critical Ramsey numbers," *Discrete Appl. Math.,* vol. 159, 2011, pp. 328–334.

[7] G. Károlyi and V. Rosta, "Generalized and geometric Ramsey numbers for cycles," *Theoret. Comput. Sci.,* vol. 263, 2001, pp. 87–98.

[8] B. Li, B. Ning, "Exact bipartite Turán numbers of large even cycles," *Journal of Graph Theory,* vol. 97(4), 2021, pp. 642–656.

[9] Y. Liu, Y. Chen, "Star-critical Ramsey Numbers of Wheels Versus Odd Cycles," *Acta Mathematicae Applicatae Sinica, English Series,* vol. 38(4), 2022, pp. 916–924.

[10] S. P. Radziszowski, "Small Ramsey numbers," *Electron. J. Combin.,* 2021, DS1.16.

[11] V. Rosta, "Ramsey Theory Applications," *Electronic Journal of Combinatorics*, 2004, Dynamic Survey 13.

[12] M. Snir, "On parallel searching," *SIAM Journal Computing,* vol. 14, 1985, pp. 688–708.

[13] D. R. Woodall, "Maximal circuits of graphs I," *Acta Math. Acad. Sci. Hungar.,* vol. 28, 1976, pp. 77–80.

[14] Y.L. Wu, Y.Q. Sun, S. Radziszowski, "Wheel and star-critical Ramsey numbers for quadrilateral," *Discrete Appl. Math.,* vol. 186, 2015, pp. 260–271.

[15] Y. Yuansheng, P. Rowlison, "On graphs without 6-cycles and related Ramsey numbers," *Utilitas Mathematica,* vol. 44, 1993, pp. 192–196.

[16] Y. Zhang, H. Broersma, Y. Chen, "On star-critical and upper size Ramsey numbers," *Discrete Appl. Math.,* vol. 202, 2016, pp. 174–180.